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Topology and renormalisability: II

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Abstract. The connection between space–times related by a covering projection and the renormalisability of quantum field theories defined on them is re-examined in the framework of constructive field theory. It is shown that a previous derivation is the heuristic version of a conditioning argument and the possibility of applying such an argument to infinite fundamental group situations (in the specific guise of finite-temperature field theory) and to boundary-value problems is explored. It is found that the implications of conditioning are broadly (though not in every detail) in agreement with known facts from renormalised perturbation theory.

1. Introduction

In a previous publication (Banach 1980b) the author showed that a quantum field theory on a multiply connected background could be derived from the corresponding field theory on the universal cover of the background by integrating out certain degrees of freedom of the covering space theory. This was done by an explicit integration of the relevant variables of the external source in the vacuum generating functional $Z(J)$. The consequences of this were the following: if the covering space $Z(J)$ had already been made finite, by appropriate renormalisation of the relevant constants in the action functional, then the resulting $Z(J)$ for the multiply connected background would automatically be finite since the integration is independent of these constants. This means that the renormalisation procedures for the two theories are essentially identical.

The simplicity of this reasoning is extremely appealing, but is the argument in fact correct? It is to this and related questions that this paper is addressed.

A number of problems and difficulties suggest themselves quite readily. Firstly, the argument presented in Banach (1980b) was for a finite fundamental group ($|\Gamma| < \infty$) only; can it be extended in any reasonable way to the $|\Gamma| = \infty$ situation? On the other hand, regardless of the size of $|\Gamma|$, the multiply connected problem can be represented as a boundary-value problem on the universal cover by choosing some suitable fundamental domain, and imposing appropriate periodicity conditions on the boundaries. Does the method yield a reasonable treatment of this version of the problem, and in fact of boundary-value problems in general? Here there is an immediate

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difficulty. The argument as originally presented (and briefly recapitulated in § 2) would suggest that all one needs to do in a boundary-value problem is to integrate out those functions in the function space from which J takes its values, which are supported in the exterior of the domain. This unfortunately does not give us any clue about what boundary conditions might arise as a result of such a procedure or indeed how we might implement any particular boundary or periodicity conditions we might be interested in. This, of course, is a serious drawback as different boundary conditions can give quite different physical properties to a given system.

In quantum field theory, different boundary conditions are intimately related to different self-adjoint extensions of a core Laplacian in a region. Thus, if our functional integral method is to be believed, it must be refined at least to the point where self-adjointness emerges in some natural way. Here, however, there is a profound difficulty. In conventional perturbation theory functional integrals serve little purpose other than combinatorial housekeeping, which they do with such remarkable efficiency that we usually forget that the conventional manipulations that we subject them to have no mathematical validity whatsoever and are, in reality, only an astonishing shorthand for manipulations of much greater complexity on the basic elements of interaction picture field theory (which, admittedly, have an equally dubious mathematical pedigree). To accomplish the above objective, then, we must at least go to a framework where functional integrals *do* have some mathematical validity, namely the scenario of measures on function spaces defined by generalised random processes in Euclidean constructive field theory, whereupon another problem presents itself. For the case of four-dimensional space–time, no non-trivial quantum field theories are known to exist—none have been constructed explicitly, so it seems that in the most physically interesting case, the only mathematically acceptable framework for treating our problem is incapable of even addressing it.

Accordingly, we will set our sights somewhat lower and will propose a generalisation of the ideas of Banach (1980b) which is applicable to cut-off field theories. Of course this begs a lot of questions since it is the no cut-off limit that is of most interest to us, but even with these restrictions, we will find that the argument fails in the $|\Gamma| = \infty$ case.

Nevertheless, all is not lost. The original conclusion of Banach (1980b) is still valid and proofs for the $|\Gamma| < \infty$ case and for finite-temperature field theory ($|\Gamma| = \infty$) are given in § 2. These essentially follow Tyutin (1981) and, being of BHPZ type, avoid functional integrals and their horrendous problems altogether.

The plan of this paper is as follows. In § 2 we will briefly recapitulate the argument of Banach (1980b) and give the BHPZ proofs referred to. The section closes with an outline, in non-mathematical terms, of what is to follow. In § 3 we set up the apparatus of generalised random processes and of functional integrals, for which we will make use of two of the many specific models available. We will also show that the argument of Banach (1980b) is a non-rigorous version of a conditioning argument which we describe.

In §§ 4 and 5 we see how this argument fares when confronted with known data from renormalised perturbation theory. In § 4 we examine the multiply connected case and we see that in the $|\Gamma| < \infty$ case everything is fine while if $|\Gamma| = \infty$ the conditioning argument breaks down. In § 5 we treat boundary-value problems and we see that in broad terms renormalisability correlates well with conditioning.

Section 6 is a conclusion wherein we make some suggestions as to natural follow ups to this work. Three appendices contain technical details.

2. Topology and renormalisability

The *sine qua non* of this paper is the following derivation. Let us write the vacuum generating functional on a multiply connected space M in the automorphic representation on its universal cover $\tilde{M}$,

$$Z^a(J)=\int [\mathrm{d}\phi][\mathrm{d}\lambda]\exp\{\mathrm{i}\hbar^{-1}[S(\phi)+\hbar\lambda(\phi-\phi^a)+J\phi]\} \tag{1}$$

where the integration over the auxiliary field λ is just a delta functional to restrict the integration over ϕ to only those ϕ satisfying the periodicity condition appropriate to the identification of the process with one on M, which we denote by the label a. By the results of Banach (1980a) there is an orthogonal split in $L^2(\tilde{M})$ when the fundamental group is finite (the propagators respecting this split)

$$\phi=\phi^a+\phi^\perp \qquad \phi^a\phi^\perp=0 \tag{2}$$

where, as in (1), juxtaposition of fields denotes the inner product in $L^2(\tilde{M})$. Noting now that

$$\lambda(\phi-\phi^a)=\lambda\phi^\perp=\lambda^\perp\phi^\perp=\lambda^\perp\phi \tag{3}$$

$$[\mathrm{d}\lambda]=[\mathrm{d}\lambda^a][\mathrm{d}\lambda^\perp] \tag{4}$$

we can do the λ integration getting

$$Z^a(J)=\int [\mathrm{d}\lambda^a]\int [\mathrm{d}\lambda^\perp]\exp(\hbar\lambda^\perp\delta/\delta J)Z(J) \tag{5}$$

where $Z(J)$ is the vacuum generating functional on $\tilde{M}$. Removing the λ^a integration we find

$$Z^a(J)=\int [\mathrm{d}J^\perp]Z(J^a,J^\perp). \tag{6}$$

This is the main result of Banach (1980b) and shows that (at this heuristic level) the two vacuum generating functionals are related by a simple integration. The implications for renormalisability, namely that if $Z(J)$ has already been made finite then $Z^a(J)$ will not diverge, are evident.

We will now give an independent verification of this fact along conventional lines. First of all, the result is no longer novel. The case of finite-temperature field theory, which we can identify with Euclidean field theory having Euclidean time compactified to a circle (consequently having normal Euclidean field theory on $\mathbb{R}^4$ as universal cover) has received some attention (Kislinger and Morley 1976, Tyutin 1981). In addition, the wide ranging paper of Bunch (1981) for ϕ^4 theory, where he shows that the divergences of renormalised perturbation theory depend only on local quantities, is applicable here, since any multiply connected space and its universal cover are locally identical by construction, implying that the renormalisation problem is the same in either case.

We begin with the $|\Gamma|<\infty$ (space-like slice of $\tilde{M}$ compact) case and for the rest of the paper we will be exclusively concerned with the Euclidean regime. The propagator for a massive scalar field on $\tilde{M}$ will have the general form

$$\Delta(x,x')=\sum_n \frac{|\psi_n(x)\rangle\langle\psi_n(x')|}{(p_n^2+m^2)} \tag{7}$$

where the p_n^2 are the eigenvalues of the Laplace–Beltrami operator on $\tilde{M}$ and the summation in (7) is over a discrete set of p_n^2 with the allowance for a single integration over a frequency variable (we assume the time variable is not compact). The integral for some Feynman diagram in the perturbation expansion will have the following form—where we highlight the structure of one particular vertex

$$\ldots \int_{\tilde{M}} \mathrm{d}\mu_{x'} \sum_{n_a} \frac{|\cdot\rangle\langle\psi_{n_a}(x')|}{(p_{n_a}^2+m^2)} \ldots \sum_{n_l} \frac{|\cdot\rangle\langle\psi_{n_l}(x')|}{(p_{n_l}^2+m^2)} \sum_{n_\alpha} \frac{|\psi_{n_\alpha}(x')\rangle\langle\cdot|}{(p_{n_\alpha}^2+m^2)} \ldots \sum_{n_\lambda} \frac{|\psi_{n_\lambda}(x')\rangle\langle\cdot|}{(p_{n_\lambda}^2+m^2)} + \mathrm{CT}$$

$$= \sum_{n_a} \ldots \sum_{n_l} \sum_{n_\alpha} \ldots \sum_{n_\lambda} \left(\frac{|\cdot\rangle}{(p_{n_a}^2+m^2)} \ldots \frac{|\cdot\rangle}{(p_{n_l}^2+m^2)} \right.$$

$$\left. \times C^{n_a \ldots n_l}_{n_\alpha \ldots n_\lambda} \frac{\langle\cdot|}{(p_{n_\alpha}^2+m^2)} \ldots \frac{\langle\cdot|}{(p_{n_\lambda}^2+m^2)} + \mathrm{CT} \right) \tag{8}$$

where

$$C^{n_a \ldots n_l}_{n_\alpha \ldots n_\lambda} = \int_{\tilde{M}} \mathrm{d}\mu_{x'} \langle\psi_{n_a}(x')| \ldots \langle\psi_{n_l}(x')|\,|\psi_{n_\alpha}(x')\rangle \ldots |\psi_{n_\lambda}(x')\rangle. \tag{9}$$

The $C^{n_a \ldots n_l}_{n_\alpha \ldots n_\lambda}$ coefficients are the generalisation to space–times of a compact space-like section of momentum conserving delta functions in flat-space Feynman diagram expansions and the analogues of the momentum integrations are the summations over the indices n.

We now assume that the theory is renormalisable on $\tilde{M}$ so that it is possible to choose subtractions (labelled CT in (8)) to render all diagrams consistently finite. How such subtractions might be achieved is a matter for speculation. Certainly the Taylor series methods of conventional BHPZ theory break down, since we are dealing with a discrete variable, although in certain cases (e.g. finite-temperature theory to be described below) it may be possible to interpolate analytically the momentum variable between its discrete values and thus to apply Zimmerman's $-t_p^{d(\gamma)}$ operators. Where this is impossible, other subtraction prescriptions such as zeta-function regularisation or dimensional regularisation come to mind. By assumption, such problems are behind us.

We now recall that the fact that $\tilde{M}$ is a covering space for M implies that it is possible to choose a basis for the spectral decomposition (7) of the propagator on $\tilde{M}$ such that the automorphic representative of the propagator for M on $\tilde{M}$ is simply a subsum of (7) (Banach 1980a); those eigenfunctions of the Laplace–Beltrami operator which do not display the correct invariance properties under the fundamental group simply do not appear.

This allows us to immediately deduce the renormalisability of the theory on M from that on $\tilde{M}$. By the finiteness of the theory on $\tilde{M}$ all relevant sums such as (8) are absolutely convergent† so that the corresponding expressions for their automorphic analogues, being simply subsums of expressions like (8), will also converge absolutely.

† There is a slight abuse of language here. In flat space, it is the momentum integrations which after renormalisation become absolutely convergent and in this respect the integrand (divergent expresison plus subtractions) is treated as a whole without regard to its structure as a sum of (individually divergent) parts. When we say a sum converges absolutely we mean that the moduli of individual terms may be summed in an arbitrary manner. This cannot be true of (8). Rather, we must treat the summand as a whole, whereupon the moduli of complete $\{n_a \ldots n_l, n_\alpha \ldots n_\lambda\}$ contributions (indicated by large round brackets in (8)) can be summed arbitrarily if the subtractions have been done correctly. We retain the use of the 'absolutely convergent' nomenclature for consistency with flat-space conventions for the purposes of this section only.

Having disposed of the $|\Gamma| < \infty$ case we will now consider finite-temperature field theory, a specific example of the $|\Gamma| = \infty$ case. A typical diagram in the covering theory will produce an expression of the following form where we highlight some particular momentum integration

$$\ldots \int d^4 p_i \left(\frac{P(q_\alpha)}{(q_a^2 + m^2)(q_b^2 + m^2) \ldots (q_l^2 + m^2)} \ldots + \text{CT} \right). \tag{10}$$

Here the q_i and q_α are linear combinations of p_i and other momenta, and $P(q_\alpha)$ is a polynomial. The finite-temperature version of this is

$$\ldots \frac{2\pi}{\beta} \sum_{n_i} \int d^3 p_i \left(\frac{P(\hat{q}_\alpha)}{(\hat{q}_a^2 + m^2)(\hat{q}_b^2 + m^2) \ldots (\hat{q}_l^2 + m^2)} + \widehat{\text{CT}} \right) \tag{11}$$

where $\hat{q}_a$ is q_a with p_{io} replaced everywhere by $2\pi n_i/\beta$ and $\widehat{\text{CT}}$ is the result of making a similar replacement in CT (an example of the analytic interpolation mentioned above†). Now the integrand in (10) is a rational function which is absolutely integrable. Therefore it must decrease sufficiently rapidly beyond its outermost singularity in momentum space, and will have convex modulus. In this region the integral absolutely dominates the sum (11) and so causes the large $|n_i|$ part of the sum to converge. The small $|n_i|$ part of the sum converges by the observation that all the singularities of the integrand are non-real. So (1) converges and finite-temperature field theory is renormalisable.

For more general $|\Gamma| = \infty$ situations we can speak with less certainty. It is clear that we will generally have to deal with the transition from a continuous to a discrete spectrum so we will be trying to deduce the finiteness of $\Sigma_n f(\lambda_n)\mu(n)$ from that of $\int d\nu(x) f(x)$. Unfortunately the convergence of either of these does not imply the convergence of the other in the absence of strong restrictions—even for entire functions $f(x)$‡—so that without exploring the structure of the given space–time and its propagators in detail, we cannot assert renormalisability. One's feeling, however, is that such space–time structure would have to be extremely wild in some sense for renormalisability to break down.

This completes our renormalisability proofs apart from two further points. For the $|\Gamma| < \infty$ case we proved renormalisability of the automorphic version of the theory in which there are $|\Gamma|$ copies of the theory present on $\tilde{M}$, one in each translate of the fundamental domain, by a fundamental group element. This differs from the theory on M proper by finite rescalings of volume integrals and of propagators. Since the rescalings are finite, a finite renormalisation will take us from one version to the other and hence will not affect renormalisability. This is made even more plausible in the functional integral derivation since we can move between the two versions simply by adjusting the value of $\hbar$ in (1) by a factor of $|\Gamma|$. For $|\Gamma| = \infty$ the sitation is qualitatively different, as the automorphic version of the theory would involve an infinite number of copies of the theories coexisting on $\tilde{M}$ and so would yield divergent answers even if each copy was finite. Accordingly, in going from (10) to (11) we went directly to the theory on M, i.e. the cylinder in our case, avoiding the problems of the automorphic version. This qualitative difference will emerge in § 4 in another guise.

† For a proof that this procedure is well defined, see appendix 1.

‡ Consider $\int f(x)\,dx$ and $\Sigma_n f(n)$ for the following two entire functions:

$$f_1(x) = x^2 \sin^2 \pi x$$

$$f_2(x) = \sum_{n=-\infty}^{+\infty} \frac{|n|!}{2^{|n|}} \phi(|n|!(x-n)) \qquad \text{with } \phi(x) = (2\pi)^{-1/2} \exp(-\tfrac{1}{2}x^2).$$

The second point concerns the dependence of the counterterms on any dimensional parameters defining a fundamental domain on $\tilde{M}$. The functional derivation suggests, and the analysis of Bunch (1981 and references therein) for ϕ^4 theory confirms, that the counterterms may be chosen to be independent of such global quantities. This is evident in our treatment of the $|\Gamma| < \infty$ case since no such parameters are introduced in the passage from the covering to the automorphic theory (their presence being implicit in the spectral resolutions), and the finite renormalisation involved in the transformation to the theory on M involves only $|\Gamma|$, an integer. In the $|\Gamma| = \infty$ case our treatment does not make this fact obvious since β appears explicitly. It is, nevertheless, the case that a finite renormalisation can remove any β dependence as shown by Tyutin (1981).

We will now describe the remainder of the paper, which is of a rather more mathematical character than hitherto.

Our attempt to mathematicise the functional derivation focuses on (6) and (1). The following fact is obvious. Given an integrable function of two variables $Z(J^a, J^\perp)$ whose total integral is finite, the integral over one of them will be an AE finite function of the other (this is just Fubini's theorem). Thus if we can express the vacuum generating functional in quantum field theory as an integrable function in a mathematically valid way, the renormalisability proof will go through. There is a difficulty here though. Even quite innocent looking functional integrands can have unacceptable behaviour when an attempt is made to integrate them so that there is no guarantee that any $Z(J)$ is integrable even if it can be shown to exist.

On the other hand, there is a rigorous way of introducing the delta functional in (1) if the integral there is sufficiently well behaved. It is the method of conditioning which effectively restricts the degrees of freedom in the integrand to the ones required. Thus what can be done is the following.

Gaussian integrals (free Euclidean fields) can be handled easily and it can be shown that a free field B can be obtained by conditioning ('integrating out' the relevant degrees of freedom) from a free field A if and only if $\Delta G = G_A - G_B$ (where G_X is the Euclidean propagator for the field X) is positive semidefinite (i.e. $(f, \Delta G f) \geq 0$ for all suitable f). Thus we might hope that interacting field theory being a hopefully 'small' perturbation of free field theory would not disturb this situation too much and that $G_A - G_B$ positive semidefinite would imply that interacting theory B could be conditioned out of interacting theory A. This is, generally speaking, borne out by our findings. However, while we find that when the renormalisation algorithm for theory A is sufficient to renormalise theory B, then $G_A - G_B \geq 0$, the converse is not true; for in the certain knowledge that theory A is insufficient to renormalise theory B we occasionally find that, nevertheless, $KG_A - G_B \geq 0$ for some finite $K \geq 1$. Were the connection between conditioning and renormalisability as simple as suggested above, this would imply that a finite rescaling of the propagator of A would be enough to make theory B renormalisable according to the renormalisation prescription of A, a clear contradiction. In the following sections we demonstrate the above by specific examples.

3. Generalised random processes and conditioning

It is common knowledge that $\int dx \exp(\frac{1}{2}ix^2)$ does not converge properly at infinity and that this leads to great difficulties in constructing a Feynman path integral which is mathematically valid and has a useful amount of predictive power. The same cannot be said for $\int dx \exp(-\frac{1}{2}x^2)$ whose behaviour is impeccable. This latter integral, correctly normalised, can thus be multiplied by itself at will and gives the inspiration for the construction of Euclidean function integrals of Gaussian type. Because of the resemblance of the resulting expressions to what one finds in statistical mechanics and probability theory, the language of stochastic processes is commonly used. More importantly, the equivalence theorem of Osterwalder and Schrader (1973, 1975) shows that properly constructed Euclidean Green functions (Schwinger functions) are the analytically continued Minkowski Green functions of a corresponding Minkowski space quantum field theory, so that we lose nothing in studying the Euclidean version of a theory (most of our space–times from now on will be, at worst, simple subsets of Minkowski space for the sake of the added mathematical precision such a restriction enables us to attain). Below we outline those aspects of Gaussian random processes we will subsequently need. For more details see Gross (1964), Simon (1974, 1979), Gelfand and Vilenkin (1964), Minlos (1963), Yosida (1968) and Halmos (1950).

A random variable is a map from a set X to the reals, which is measurable in some σ-algebra $\mathscr{F}$ of subsets of X which, together with a positive measure μ of unit total mass, turns X into a probability space. A random process is a map from some parametrising set T to random variables on X. A generalised random process is a random process for which the parametrising set is itself a real locally convex topological vector space $\mathscr{T}$. An automorphic process is one for which $\mathscr{T}$ consists of functions satisfying an automorphic condition (25).

Let us denote the generalised random process by Φ, which from now on will be linear; $\Phi(\tau_1+\alpha\tau_2)=\Phi(\tau_1)+\alpha\Phi(\tau_2)$. Then if $\tau_1\ldots\tau_n\in\mathscr{T}$, the expectations $E((\Phi(\tau_1),\ldots,\Phi(\tau_n))\in A\subset\mathbb{R}^n)=\mu((\Phi(\tau_1),\ldots,\Phi(\tau_n))^{-1}(A))$ for measurable $A\subset\mathbb{R}^n$, define the joint probability distributions of the $\{\Phi(\tau_i)\}$ and satisfy the consistency condition that if $\{\tau_{i_1}\ldots\tau_{i_m}\})$ is a subset of $\{\tau_1\ldots\tau_n\}$ then $E((\Phi(\tau_{i_1}),\ldots,\Phi(\tau_{i_m}))\in A\subset\mathbb{R}^m)=E((\Phi(\tau_1),\ldots,\Phi(\tau_n))\in A\times\mathbb{R}^{n-m}\subset\mathbb{R}^n)$. The joint probability distributions define an integral for tame functions on $\mathscr{T}'$, the dual space of $\mathscr{T}$. A tame function on $\mathscr{T}'$ is a bounded continuous map to the complex numbers, constant on the cosets of $\mathscr{T}'$ under the equivalence relation $(F_i\in\mathscr{T}')$, $F_1\sim F_2\Leftrightarrow((F_1,\tau_1),\ldots,(F_1,\tau_n))=((F_2,\tau_1),\ldots,(F_2,\tau_n))$ for some finite set $\{\tau_i\}\subset\mathscr{T}$. Such a function f induces in the obvious way a function $\hat{f}$ on $\mathscr{T}'/\mathscr{N}(\tau_1\ldots\tau_n)$ (with its natural $\mathbb{R}^n$ structure), where $\mathscr{N}(\tau_1\ldots\tau_n)$ is the annihilator of the subspace of $\mathscr{T}$ generated by $\{\tau_1\ldots\tau_n\}$, and the proclaimed integral is defined by

$$I(f)=\int_{\mathbb{R}^n}\hat{f}\,\mathrm{d}E((\Phi(\tau_1)\ldots\Phi(\tau_n))\in\mathbb{R}^n). \tag{12}$$

Conversely, given such a family of consistent joint probability distributions (which is the normal starting point in practice) we can construct the generalised random process from which they derive. In particular we can do so by extending the integral (12) from tame functions to a more general class of functions on an underlying probability space as the following version of the Kolmogorov extension theorem shows. The proof is essentially an argument of Gross (1964).

Theorem 1. Given an integral on tame functions on $\mathscr{T}'$ given by a family of consistent joint probability distributions parametrised by finite subsets of $\mathscr{T}$, there is a measure space $(X_1,\mathscr{F}_1,\mu_1)$ for which the joint probability distributions are those of a set of random variables on X_1 and which extends the integral on the tame functions to $L^1(X_1)$.

Proof. The tame functions with the supremum norm generate a commutative C^* algebra $\mathscr{A}$. By the Gelfand–Naimark theorem $\mathscr{A}$ is isometrically *-isomorphic to $C(X_1)$ where X_1 is the compact Hausdorff maximal ideal space of $\mathscr{A}$. The integral (12) is a positive normalised continuous linear functional on the tame functions and so extends by continuity to a positive normalised continuous linear functional on $\mathscr{A}$ which in turn is representable as a regular Borel probability measure μ_1 on X_1 by the Riesz–Markov theorem. Let $\hat{x}_n(\tau)$ be the representative on $\mathscr{T}'/\mathscr{N}(\tau) \approx \mathbb{R}$ given by $\hat{x}_n(\tau)(x) = x\phi_n(x)$, with $1 = \Sigma_n \phi_n(x)$ being a partition of unity on $\mathbb{R}$, of a tame function $x_n(\tau)$ on $\mathscr{T}'$, having image $x'_n(\tau)$ in $L^1(X_1)$. The sum $x'(\tau) = \Sigma_n x'_n(\tau)$, being a countable sum of measurable functions, is clearly a random variable on $(X_1, \mathscr{F}_1, \mu_1)$ and the joint probability distributions of the $\{x'(\tau)\}_{\tau\in\mathscr{T}}$ evidently coincide with those originally given, yielding a generalised random process $\{\Phi(\tau)\}_{\tau\in\mathscr{T}}$, completing the proof.

The above is one version, out of many possible constructions, of 'the random process indexed by $\mathscr{T}$ and given by the $\{\Phi(\tau)\}_{\tau\in\mathscr{T}}$'. It illustrates the intimate connection between random processes and functional integrals and also the fact that, in general, the functional integral thus constructed is defined on a larger space X_1 than $\mathscr{T}'$. This does not matter if all the analysis can be done using the properties of the space $\mathscr{T}$ alone, as is frequently the case when $\mathscr{T}$ is a Hilbert space. In such cases the precise nature of X_1 is largely irrelevant. In other cases (one will arise in § 4), we would really like to know more about the measure space itself. For this we can use the construction of Minlos.

Theorem 2 (Minlos). Let $\mathscr{T}$ be a nuclear space and the generalised random process given by the joint probability distributions of the $\{\Phi(\tau)\}_{\tau\in\mathscr{T}}$ make the integral on tame functions (12) weakly continuous in $\mathscr{T}$. Then there is a countably additive measure on $\mathscr{T}'$ which extends (12) to integrable functions measurable in the σ-algebra of Borel sets of the (weak) topology of $\mathscr{T}'$.

For the proof see Gelfand and Vilenkin (1964), Minlos (1963).

The above theorem shows that provided sufficient restrictions are imposed on $\mathscr{T}$ and on Φ, we need not go beyond $\mathscr{T}'$ to construct our functional integral.

Gaussian generalised random processes are given by positive quadratic forms on $\mathscr{T}$. Thus if $C(\tau, \rho)$ is the bilinear form on $\mathscr{T}$ of a positive quadratic form, then $\{\Phi(\tau)\}_{\tau\in\mathscr{T}}$ are Gaussianly distributed with covariance C if

$$\mu((\Phi(\tau_1)\ldots\Phi(\tau_n))\in A\subset\mathbb{R}^n) = \frac{\det[a_{ij}]^{1/2}}{(2\pi)^{n/2}}\int_A \mathrm{d}^n\xi\,\exp\Big(-\tfrac{1}{2}\sum_{ij} a_{ij}\xi_i\xi_j\Big) \tag{13}$$

with $a_{ij} = C(\tau_i, \tau_j)^{-1}$. Since for such a process

$$E(\Phi(\tau_1)\ldots\Phi(\tau_n)) = \int \Phi(\tau_1)\ldots\Phi(\tau_n)\,\mathrm{d}\mu = \sum_{\text{pairings}} C(\tau_{i_1}, \tau_{i_2})\ldots C(\tau_{i_{n-1}}, \tau_{i_n}) \tag{14}$$

specifying C is equivalent to specifying the whole process.

We now introduce the idea of conditioning. Let $\mathscr{G}$ be a σ-subalgebra of $\mathscr{F}$ in the probability space $(X, \mathscr{F}, \mu)$. We define the conditional expectation of $f\in L^1$, with respect to $\mathscr{G}$, $E(f|\mathscr{G})$, as the $\mathscr{G}$-measurable random variable satisfying

$$\int fg\,\mathrm{d}\mu = \int E(f|\mathscr{G})g\,\mathrm{d}\mu \tag{15}$$

for all $\mathscr{G}$-measurable g. We can construct $E(f|\mathscr{G})$ more directly by noting that for $f \in L^2$, $E(f|\mathscr{G})$ is the projection onto $L^2(X, \mathscr{G}, \mu)$ of f. Since L^2 is dense in L^1 and $\|E(f|\mathscr{G})\|_1 \le \|f\|_1$ this projection extends to the whole of L^1 by continuity. If $\mathscr{G}$ is the smallest σ-subalgebra of $\mathscr{F}$ for which the random variables $\{\Phi(\tau): \tau \in T, T$ a closed subspace of $\mathscr{T}\}$ are measurable we call the $E(f|\mathscr{G})$ random variables conditioned on T.

All of the above is probability theory. We have yet to make contact with physics. This essential step was made by Nelson (1973a, b) who showed that free Euclidean field theory could be constructed as a Gaussian random process and gave a sufficient (though not necessary) set of conditions for continuing back to Minkowski space. Let $\mathscr{H}_{-1}$ be the Hilbert space of real distributions for which the inner product

$$\langle u, v\rangle_{-1} = \langle u, G_0 v\rangle_{L^2} = \langle u, (-\Delta + m^2)^{-1} v\rangle_{L^2} = \frac{1}{(2\pi)^4}\int \frac{\overline{\tilde{u}(p)}\tilde{v}(p)}{(p^2+m^2)}\,\mathrm{d}^4 p \tag{16}$$

converges. The free Markov scalar field is defined as the Gaussian random process indexed by $\mathscr{H}_{-1}$ with covariance given by the inner product (16). According to what was said above this is sufficient to specify the free field as a functional integral. An alternative construction, employing Minlos's theorem, would give the free field as the Gaussian random process on $\mathscr{S}'$ (the space of real tempered distributions) with covariance given by the same inner product (16) except that we restrict the range of u and v to $\mathscr{S}$, the real smooth functions of rapid decrease. Since (16) is continuous in u and v in the nuclear topology of $\mathscr{S}$, the integral on tame functions it defines is weakly continuous and the conditions of Minlos's theorem are satisfied. We will use either of these constructions as convenient in the sequel.

Both of the constructions possess many important properties vital in the construction of non-trivial field theories; however, we will primarily need only one, common to both constructions and contained in the following circle of ideas invented by Guerra *et al* (1975, 1976, hereafter referred to as GRSI and GRSII).

Suppose $\mathscr{T}$ is a Hilbert space and let H be a closed subspace so that $\mathscr{T} = H \oplus H^{\perp}$ and consider the Gaussian random process indexed by $\mathscr{T}$ with covariance given by the inner product $\langle\ ,\ \rangle$ on $\mathscr{T}$. Then if $\{h_1 \dots h_n\} \in H$ and $\{h_1^{\perp} \dots h_m^{\perp}\} \in H^{\perp}$ we find

$$\begin{aligned}\mu((\Phi(h_1), \dots, \Phi(h_n), \Phi(h_1^{\perp}), \dots, \Phi(h_m^{\perp})) \in A \times B; A \subset \mathbb{R}^n, B \subset \mathbb{R}^m)\\ = \mu((\Phi(h_1), \dots, \Phi(h_n)) \in A)\mu((\Phi(h_1^{\perp}), \dots, \Phi(h_m^{\perp})) \in B)\end{aligned} \tag{17}$$

and so

$$\begin{aligned}E(\Phi(h_1)\dots\Phi(h_n)\Phi(h_1^{\perp})\dots\Phi(h_m^{\perp}))\\ = \int \Phi(h_1)\dots\Phi(h_n)\Phi(h_1^{\perp})\dots\Phi(h_m^{\perp})\,\mathrm{d}\mu\\ = \Big(\sum_{\text{pairings}} \langle h_{i_1}, h_{i_2}\rangle \dots \langle h_{i_{n-1}}, h_{i_n}\rangle\Big)\Big(\sum_{\text{pairings}} \langle h_{j_1}^{\perp}, h_{j_2}^{\perp}\rangle \dots \langle h_{j_{m-1}}^{\perp}, h_{j_m}^{\perp}\rangle\Big)\\ = \int \Phi(h_1)\dots\Phi(h_n)\,\mathrm{d}\mu_H \int \Phi(h_1^{\perp})\dots\Phi(h_m^{\perp})\,\mathrm{d}\mu_{H^{\perp}}\end{aligned} \tag{18}$$

where the last expression in (18) is the expectation in the product probability space $(X_1 \times X_2, \mathscr{F}_1 \otimes \mathscr{F}_2, \mathrm{d}\mu_H \otimes \mathrm{d}\mu_{H^{\perp}})$ formed from the probability spaces of the Gaussian random processes indexed by H and $H^{\perp}$, being the underlying probability space of the Gaussian random process indexed by $H \oplus H^{\perp}$. Thus an orthogonal decomposition of $\mathscr{T}$ induces a split in the measure

$$\mathrm{d}\mu_{\mathscr{T}} = \mathrm{d}\mu_H \otimes \mathrm{d}\mu_{H^\perp}. \tag{19}$$

On the other hand, given such a split in the measure and $f \in L^1$ we find

$$\int f \, \mathrm{d}\mu_{H^\perp} = E(f|\mathscr{F}_1 \otimes X_2) \tag{20}$$

since the left-hand side of (20) clearly satisfies (15) for all $\mathscr{F}_1 \otimes X_2$-measurable g. Thus when there is a split, random variables conditioned on one subspace are obtained by integrating out the degrees of freedom indexed by the other. More generally, given a split in the probability space $X = X_1 \times X_2$, $\mathrm{d}\mu = \mathrm{d}\mu_1 \otimes \mathrm{d}\mu_2$ not necessarily subordinate to a split in an indexing space, equation (20) (with the subscript $H^\perp$ replaced by 2) remains a valid definition of the conditional expectation. The circumstances under which such a generalised split occurs for Gaussian random processes are delineated in the following fundamental result of GRSI.

Theorem 3. Let $\mathscr{T}$ be a real locally convex space and let ϕ_1 and ϕ_2 be Gaussian random processes indexed by $\mathscr{T}$ with covariances C_1 and C_2. Then ϕ_2 can be obtained from ϕ_1 by generalised conditioning if and only if $\Delta C = C_1 - C_2$ is positive semidefinite.

Proof. If ϕ_2 arises from ϕ_1 by generalised conditioning then (since this is a projection) we have

$$\begin{aligned} 0 \leqslant C_2(\tau, \tau) &= \int \phi_2^2(\tau) \, \mathrm{d}\mu^{(2)} = \int E^2(\phi_1(\tau)|\mathscr{G}) \, \mathrm{d}\mu^{(1)} = \|E(\phi_1(\tau)|\mathscr{G})\|_2^2 \\ &\leqslant \|\phi_1(\tau)\|_2^2 = \int \phi_1^2(\tau) \, \mathrm{d}\mu^{(1)} = C_1(\tau, \tau) \end{aligned} \tag{21}$$

so that $\Delta C \geqslant 0$. Conversely if $\Delta C \geqslant 0$ and $C_2 \geqslant 0$ then the Gaussian random process $\Phi(\tau) = \phi_2(\tau) + \phi_\Delta(\tau)$ indexed by $\mathscr{T}$ with covariance $C_1 = C_2 + \Delta C$ can be realised as the product process $(X_2 \times X_\Delta, \mathscr{F}_2 \otimes \mathscr{F}_\Delta, \mathrm{d}\mu_2 \otimes \mathrm{d}\mu_\Delta)$ where $(X_\Delta, \mathscr{F}_\Delta, \mathrm{d}\mu_\Delta)$ is the probability space of the process given by ΔC, completing the proof.

We have now assembled enough apparatus to be able to make a comparison with the heuristic derivation of equations (1)–(6). First we rewrite $Z(J)$ in a meaningful Euclidean form:

$$Z(J) = \int_X \mathrm{d}\mu \, \exp(-S_I^c(\Phi) + \mathrm{i}\Phi(J)). \tag{22}$$

Here $J \in \mathscr{T}$, $\mathrm{d}\mu$ is the free field measure on a probability space X corresponding to a Gaussian random process Φ indexed by $\mathscr{T}$ and with covariance given by the propagator as in (16) and $S_I^c(\Phi)$ is a *cut-off* interaction which makes the measure $\exp(-S_I^c(\Phi)) \, \mathrm{d}\mu$ absolutely continuous with respect to $\mathrm{d}\mu$. Under these circumstances $Z(J)$ is the characteristic function (or Fourier transform) of the cut-off measure and is well defined since $\exp(\mathrm{i}\Phi(J))$ is measurable.

Given a direct decomposition of $\mathscr{T} = T \oplus S$ such that $J = t(J) + s(J)$ where $t(J)$ and $s(J)$ are the respective projections of J, we can define Φ_T by $\Phi_T(J) = \Phi(t(J))$. If C and C_T, C_S are respectively the covariance on $\mathscr{T}$ and its restrictions to T and S

with $C = C_T \oplus C_S$ clearly $C - C_T \geqslant 0$ and so, by theorem 3, $\mathrm{d}\mu$ splits and displays Φ_T as a conditioned random variable. Thus we can write the theory on the subspace T in two ways

$$\int_{X_T} \mathrm{d}\mu_T \exp(-S_I^c(\Phi_T)+\mathrm{i}\Phi_T(t)) = Z_T(t) = Z_T(t(J)) = \int_X \mathrm{d}\mu \exp(-S_I^c(\Phi_T)+\mathrm{i}\Phi_T(J)) \tag{23}$$

where on the left-hand side all quantities refer to objects constructed directly using T, while on the right-hand side the same symbols refer to objects obtained by conditioning in the theory constructed from $\mathcal{T}$. This is clearly the constructive version of (1), the formal insertion of the delta functional into $Z(J)$.

Now $Z(J)$ is defined on $\mathcal{T}$ which is not, according to our constructions, a measure space. Even if it were, we have no guarantee that $Z(J)$ would be measurable (although the two problems are obviously related) so we cannot implement the integration in (6) in a well defined way. However, if both of these were the case, it is evident that $Z_T(t)$ would simply be the conditional expectation of $Z(J)$ obtained by 'integrating out S'. Although such a procedure is seemingly beyond present-day technology, its formal equivalence to (1) which *is* implementable (at least at the cut-off level) and the fact that it works directly in a formal, combinational way as verified in Banach (1980b) leads one to suspect that there may be a formulation in which (6) has a mathematically sound analogue. Such an 'inversion of the Fourier transform' would be particularly pleasing in the case of nuclear spaces where it would presumably provide a dual result to Minlos's theorem.

We must now say something about the removal of cut-offs. This of course is the heart of constructive field theory and is where the real joy, or pain, of the subject abides. It is an interaction-dependent business, each tractable model posing its own difficulties, and as such is beyond the scope of the present paper in which we seek to make more generally applicable statements. However, the following is relevant to us. In (23) all cut-offs are present and so all the convergence problems their removal creates are present too. Thus we have to confirm that cut-off removal commutes with conditioning in order to infer renormalisability. In view of the remarks we are going to make below and particularly in § 5 we have no reason whatsoever to expect that this will be a trivial exercise in general. Suppose, however, that we have obtained a cut-off-free $Z(J)$; then, were (6) a reality, conditioning of the theory without cut-offs would be a simple matter of doing the requisite integration. The apparent simplicity of such a procedure may be an indication of the difficulty of constructing (6), and in any case we would still be faced with the task of checking whether or not the two conditioning procedures yielded the same result.

Returning now to the cut-off theory we have the following basic theorem, again essentially due to results in GRSI.

Theorem 4. (1) Given a direct decomposition $\mathcal{T} = T \oplus S$, $Z_T(t(J))$ as constructed in (23) is finite if $Z(J)$ exists. (2) Given two covariances C, C_1 on $\mathcal{T}$ then a cut-off interaction yielding a finite $Z(J)$ through (22) where $\mathrm{d}\mu$ has covariance C will give a finite $Z_1(J)$ by conditioning on the product process $\{\Phi(J) = \Phi_1(J) + \Phi_\Delta(J)$, $(X_1 \times X_\Delta, \mathcal{F}_1 \otimes \mathcal{F}_\Delta, \mathrm{d}\mu_1 \otimes \mathrm{d}\mu_\Delta)$, $C_1 \oplus C_\Delta\}$ where $C_\Delta = C - C_1$ if $C_\Delta \geqslant 0$.

Proof. Both results are no more than statements of the fact that conditioned L^1 random variables are integrable with respect to the original measure and so conditioning cannot make a convergent $Z(J)$ diverge. A stronger result follows from Jensen's inequality, $\exp[-S_I^c(E(\Phi|\mathcal{F}_1 \otimes X_2))] \leq E(\exp[-S_I^c(\Phi)]|\mathcal{F}_1 \otimes X_2)$ (which applies in either case by change of notation) giving, by integrating over X_1, $Z_1(0) \leq Z(0)$. The finiteness of $Z_1(J)$ then follows since $Z_1(J)$ is continuous in J, completing the proof.

The above result takes us as far as we might expect to go in search of a generally applicable result relating conditioning and renormalisability. In the limit of no cut-offs we generally lose the absolute continuity of the interacting measure with respect to the free measure and so cannot assert that the criterion on covariances given by theorem 3 is sufficient to deduce renormalisability from conditioning. Nevertheless, in the rest of the paper we explore to what extent the two are correlated by comparing the covariance criterion with known data from renormalised perturbation theory. In doing so we find some ambiguities which become less suprising in view of the following facts.

In perturbation theory, a finite rescaling of the propagator has no drastic consequences for renormalisability; each primitive divergence simply picks up a finite factor and so a finite adjustment to its counterterm restores convergence. At a simpler level, introducing a perturbation proportional to the free action in a free theory introduces no divergences and one can sum the resulting geometric progression. On the constructive level however, the same is not the case. By theorem 3, the only multiples of a covariance for which the associated Gaussian process can be conditioned out of the process defined by that covariance are less than one. Thus if $1 < K < \infty$ and we have $KG_A \geq G_B$, $G_A \not\geq G_B$, the constructive point of view would say that the renormalisations for the theories given by G_A and G_B are inequivalent while the perturbation theoretical view would be that they should be equivalent modulo finite rescalings.

4. Multiply connected spaces re-examined

In this section and the next, we apply the criteria deduced in § 3 to specific examples. First of all, we look at multiply connected spaces. In the $|\Gamma| < \infty$ case suppose space–time is static, of the form $\tilde{M} = \tilde{\Sigma} \times \mathbb{R}$ where $\tilde{\Sigma}$ is the compact universal cover of a space-like slice of $M = \Sigma \times \mathbb{R}$ with $\mathbb{R}$ denoting here the time axis. We alter the signature of the metric to turn $\tilde{M}$ into a Riemannian space and denote by Δ the corresponding Laplace–Beltrami operator on $C_0^\infty(\tilde{M})$. For m^2 sufficiently large $(-\Delta + m^2)$ will be a positive operator in the L^2 inner product and will have inverse G. We denote by $\mathcal{H}_{-1}(\tilde{M})$ the completion of $C_{0\mathbb{R}}^\infty(\tilde{M})$ (the real smooth functions of compact support) in the inner product

$$\langle u, v\rangle_{-1} \equiv \langle u, Gv\rangle_{L^2} \tag{24}$$

and define the Euclidean scalar field of mass m on $\tilde{M}$ as the Gaussian random process indexed by $\mathcal{H}_{-1}(\tilde{M})$.

Let $\{a(\gamma)\colon \gamma \in \Gamma\}$ be a real representation of the fundamental group of Σ and let $\mathcal{H}_{-1}^a(\tilde{M})$ be the subspace of $\mathcal{H}_{-1}(\tilde{M})$ whose members satisfy

$$f(\gamma x) = a(\gamma) f(x) \qquad \forall \gamma \in \Gamma, \forall x \in \tilde{M}. \tag{25}$$

The projection P_a onto this subspace is given by

$$P_a f(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} a(\gamma^{-1}) f(\gamma x). \tag{26}$$

P_a is well defined since $|\Gamma|<\infty$ and so $\mathscr{H}^a_{-1}(\tilde{M})$ is clearly non-empty. We define the automorphic version of the theory on M as the automorphic process given by the process above conditioned on the subspace $\mathscr{H}^a_{-1}(\tilde{M})$. That the covariance criterion is satisfied, follows from the general theory on conditioned Gaussian processes given in § 3, but we can see it more directly using the spectral representation of G. The spectrum of G is as described following (7) so we find

$$\langle f, Gf\rangle_{L^2}=\sum_n \frac{|f_n|^2}{(p_n^2+m^2)} \tag{27}$$

where the $\{f_n\}$ are the Fourier components of f in the basis of generalised eigenvectors of G and the summation includes the frequency integration. The automorphic covariance G_a is simply given by selecting certain of the generalised eigenvectors and so $\langle f, G_a f\rangle_{L^2}$ is given by a subsum of (27). Since all the terms in (27) are positive we thus have $G\geqslant G_a$ and the renormalisability demonstrated in § 2 is reflected in the relation between the covariances.

We may point out in passing that our construction has resolved the self-adjointness problems raised in § 1 by remote control. For we note that the projection P_a can be written as an averaged finite sum of unitary operators U^a, unitary in both $L^2(\tilde{M})$ by the invariance of the metric under the fundamental group, and in $\mathscr{H}_{-1}(\tilde{M})$ by the invariance of the Laplace–Beltrami operator,

$$P_a=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} U^a(\gamma). \tag{28}$$

If in addition $\mathscr{D}$ is the domain of self-adjointness of $(-\Delta+m^2)$ in $L^2(\tilde{M})$ we also have $U^a(\gamma):\mathscr{D}\to\mathscr{D}$ and $U^a(\gamma)(-\Delta+m^2)=(-\Delta+m^2)U^a(\gamma)$ on $\mathscr{D}$. Thus $P_a(-\Delta+m^2)=(-\Delta+m^2)P_a:\mathscr{D}_a\to L^2_a(\tilde{M})$ where $\mathscr{D}_a=P_a\mathscr{D}$ and $L^2_a(\tilde{M})=P_aL^2(\tilde{M})$. Since P_a is a projection, $(-\Delta+m^2)P_a$ is evidently self-adjoint on $\mathscr{D}_a\subset L^2_a(\tilde{M})$.

This completes our discussion of the $|\Gamma|<\infty$ case and we note that the whole derivation was performed using only the Hilbert space properties of the indexing space of the Gaussian process, displaying the usefulness of the abstract construction of theorem 1. In the $|\Gamma|=\infty$ case this is no longer possible as any function on $\tilde{M}$ satisfying (25) could not possibly belong to $L^2(\tilde{M})$ or $\mathscr{H}_{-1}(\tilde{M})$—equally, (26) is quite meaningless in this context.

To see what might be possible in the $|\Gamma|=\infty$ case we will look at finite-temperature field theory again. Here $\tilde{M}=\mathbb{R}^4$, $M=S^1\times\mathbb{R}^3$ and Γ is a group of translations along one of the coordinate axes. The model for the free field that we use will be the Gaussian random process indexed by the real Schwartz space $\mathscr{S}(\mathbb{R}^4)$ with covariance given by the inner product (16) for $u, v\in\mathscr{S}$. The advantage of doing this is that we know that the measure space is precisely $\mathscr{S}'$.

Now $\mathscr{S}'$ contains closed subspaces $\tilde{S}'^a_\beta$ of distributions satisfying the dual of the periodicity condition (25), by which we mean

$$f(h_\gamma)=f(a(\gamma)h) \qquad \forall h\in\mathscr{S} \tag{29}$$

with $h_\gamma(x)=h(\gamma x)$, where Γ is the group of translations by integer multiples of β. Let us suppose that $\mathscr{S}'=\tilde{S}'^a_\beta\oplus\tilde{T}'^a_\beta$ where $\tilde{T}'^a_\beta$ is a complement of $\tilde{S}'^a_\beta$ and the direct sum is continuous in the usual weak*-topology of $\mathscr{S}'$. Then if we condition the process on the σ-subalgebra $\mathscr{F}^a_\beta\otimes\tilde{T}'^a_\beta$ where $\mathscr{F}^a_\beta$ is the σ-algebra of Borel sets in $\tilde{S}'^a_\beta$ we should be able to achieve our objective of obtaining the automorphic theory by the following procedure.

We have $\mathscr{S}' = \tilde{S}'^{a}_{\beta} \oplus \tilde{T}'^{a}_{\beta}$. Now $\langle \mathscr{S}, \mathscr{S}' \rangle$ is a dual pair so by § 20.5 of Köthe (1969) we have that $\mathscr{S} = \tilde{T}^{a}_{\beta} \oplus \tilde{S}^{a}_{\beta}$ where $\tilde{T}^{a}_{\beta}$ is the annihilator of $\tilde{S}'^{a}_{\beta}$.

Lemma 5. $\tilde{T}^{a}_{\beta} = \{h : \Sigma_{\gamma \in \Gamma} a(\gamma^{-1}) h(\gamma x) = 0\}$.

Proof. We know that $\forall x, \Sigma_{\gamma \in \Gamma} \delta_{\gamma x} a(\gamma) = \Sigma_n (\pm)^n \delta(x_0 - n\beta) \delta_{x_1} \delta_{x_2} \delta_{x_3} \in \tilde{S}'^{a}_{\beta}$ where δ_ξ is the delta distribution at ξ; applying it to $h \in \tilde{T}^{a}_{\beta}$ shows necessity. On the other hand, if $f \in \tilde{S}'^{a}_{\beta}$, any weak*-neighbourhood of f contains a C^∞ function f_i satisfying (29) and acting as

$$f_i(h) = \int_{\mathbb{R}^4} d^4x\, f_i(x) h(x). \tag{30}$$

By countable additivity, translation invariance, (29) and dominated convergence this is equal to

$$\begin{aligned} f_i(h) &= \sum_{\gamma \in \Gamma} \int_{[0,\beta) \times \mathbb{R}^3} d^4x\, f_i(x) a(\gamma^{-1}) h(\gamma x) \\ &= \int_{[0,\beta) \times \mathbb{R}^3} d^4x\, f_i(x) \sum_{\gamma \in \Gamma} a(\gamma^{-1}) h(\gamma x) \end{aligned} \tag{31}$$

which is zero if h satisfies the hypothesis. Letting $f_i \to f$ completes the proof.

We have the following two sets of isomorphisms; $\mathscr{S}'/\tilde{T}'^{a}_{\beta} \approx \tilde{S}'^{a}_{\beta} \approx S'^{a}_{\beta}$ and $\mathscr{S}/\tilde{T}^{a}_{\beta} \approx \tilde{S}^{a}_{\beta} \approx \hat{S}^{a}_{\beta} \approx S^{a}_{\beta}$. Here $\hat{S}^{a}_{\beta}$ is the space of C^∞ automorphic functions satisfying (25), a function $\hat{h} \in \hat{S}^{a}_{\beta}$ being identified with the unique (by lemma 5) function $h \in \tilde{S}^{a}_{\beta}$ for which

$$\hat{h}(x) = \sum_{\gamma \in \Gamma} a(\gamma^{-1}) h(\gamma x). \tag{32}$$

S^{a}_{β} is the space of sections of an associated vector bundle B with base space M constructed from $\tilde{M} \times \mathbb{R}$ regarded as a product bundle via the standard construction (see appendix 2) which makes S^{a}_{β} isomorphic to the space of automorphic sections of $\tilde{M} \times \mathbb{R}$, $\hat{S}^{a}_{\beta}$; and S'^{a}_{β} is the dual space of S^{a}_{β}.

If $\tilde{\Phi}$ is the Gaussian random process indexed by $\mathscr{S}$ given above, then we can construct an automorphic process $\hat{\Phi}$ indexed by $\hat{S}^{a}_{\beta}$, with measure space $\tilde{S}'^{a}_{\beta}$; and a process Φ indexed by S^{a}_{β}, with measure space S'^{a}_{β} on the multiply connected space M, by giving the covariances

$$E(\Phi(h_{1M})\Phi(h_{2M})) = E(\hat{\Phi}(\hat{h}_1)\hat{\Phi}(\hat{h}_2)) = E(\tilde{\Phi}(h_1)\tilde{\Phi}(h_2)) \tag{33}$$

where h_{iM} is the section identified with $\hat{h}_i$, which is itself related to $h_i \in \tilde{S}^{a}_{\beta}$ by (32). Identifying $\hat{S}^{a}_{\beta}$ as a model of the dual space of $\tilde{S}'^{a}_{\beta}$, by restricting its action to a fundamental domain as in (31) and noting that the induced topologies on $\hat{S}^{a}_{\beta}$ and S^{a}_{β} are nuclear, we confirm that the measure spaces for $\hat{\Phi}$ and Φ are as given and we see that conditioning on $\mathscr{F}^{a}_{\beta} \otimes \tilde{T}'^{a}_{\beta}$ (the σ-subalgebra for which the $\{\tilde{\Phi}(h) : h \in \tilde{S}^{a}_{\beta}\}$ are measurable) yields a well defined automorphic process $\hat{\Phi}$, the pullback to the covering space of the process Φ on M, providing the covariance inequality is satisfied.

Sadly, this is not the case. To see this we must examine the complementation theory of $\tilde{S}'^{a}_{\beta}$. Since $\langle \mathscr{S}, \mathscr{S}' \rangle$ is not only a dual pair, but a reflexive pair, the complements of $\tilde{S}'^{a}_{\beta}$ are in one-to-one correspondence with complements of $\tilde{T}^{a}_{\beta}$. Let x_0 be the coordinate axis along which Γ acts and let p_0 be the corresponding Fourier transform variable. Let Q^a_β be the Fourier transform of $\tilde{T}^a_\beta$ and denote by F Fourier transformation in general.

Lemma 6. $Q^a_\beta = \{f \in F\mathcal{S}: f(p_0 = 2\pi n/\beta + \lambda(a)) = 0\}$
where

$$\lambda(a) = \begin{cases} 0 & \text{if } a \text{ is trivial} \\ \pi/\beta & \text{if } a \text{ is alternating.} \end{cases}$$

Proof. We have $\forall x \in \mathbb{R}^4$

$$\sum_{\gamma \in \Gamma} a(\gamma)\delta_{\gamma x} = \lim_{N\to\infty} \sum_{-N}^{+N} (\pm 1)^n \delta(x_0 - n\beta)\delta_{x_1}\delta_{x_2}\delta_{x_3} \tag{34}$$

as distributions. Taking the Fourier transform,

$$F\left(\lim_{N\to\infty} \sum_{-N}^{+N} (\pm 1)^n \delta(x_0 - n\beta)\delta_{x_1}\delta_{x_2}\delta_{x_3}\right) = \lim_{N\to\infty} \sum_{-N}^{+N} (\pm 1)^n F(\delta(x_0 - n\beta)\delta_{x_1}\delta_{x_2}\delta_{x_3})$$

$$= \lim_{N\to\infty} \mathrm{e}^{\mathrm{i}px} \sum_{-N}^{+N} (\pm 1)^n \,\mathrm{e}^{\mathrm{i}p_0 n\beta}$$

$$= \mathrm{e}^{\mathrm{i}px} \lim_{N\to\infty} \sum_{-N}^{+N} (\pm 1)^n \,\mathrm{e}^{\mathrm{i}p_0 n\beta}$$

$$= \mathrm{e}^{\mathrm{i}px} \lim_{N\to\infty} \frac{\sin Np_0\beta}{\phi_a(p_0)} \,\mathrm{e}^{-\mathrm{i}p_0\beta/2} \tag{35}$$

where

$$\phi_a(p_0) = \begin{cases} \sin(p_0\beta/2) & \text{if } a \text{ is trivial} \\ (-1)^N \cos(p_0\beta/2) & \text{if } a \text{ is alternating.} \end{cases} \tag{36}$$

In the limit this is equal, as a distribution, to

$$\mathrm{e}^{\mathrm{i}px}\left\{\frac{2\pi}{\beta} \sum_n \delta\left[p_0 - \left(\frac{2\pi n}{\beta} + \lambda(a)\right)\right]\right\}. \tag{37}$$

By the uniqueness of the Fourier transform, this can only annihilate $h \in F\mathcal{S}$ for all $x \in \mathbb{R}^4$ if h vanishes on the support of the factor in curly brackets. This condition is evidently also sufficient, completing the proof.

Knowing this, it is easy to construct some complements of Q^a_β. Suppose for convenience that a is trivial.

Example 7. Let $\{\phi_n(p_0), \phi'_n(p_0)\}$ satisfy the following:
(i) $0 \leqslant \phi_0 \leqslant 1$; (ii) $\operatorname{supp} \phi_0 \subset [-\frac{2}{3}\pi, \frac{2}{3}\pi]$; (iii) ϕ_0 is C^∞ real and even (and therefore in $F\mathcal{S}$); (iv) $\phi_0(0) = 1$; (v) for $n \geqslant 1$, $\phi_n(p_0) = [\phi_0(p_0 - 2\pi n/\beta) + \phi_0(p_0 + 2\pi n/\beta)]$, $\phi'_n(p_0) = \operatorname{sgn}(p_0)\phi_n(p_0)$. Then for any $h \in F\mathcal{S}$ we have

$$h(p) = \left(\sum_{n=0}^{\infty} (\operatorname{Re} h(n, \boldsymbol{p}))\phi_n(p_0) + \mathrm{i} \sum_{n=1}^{\infty} (\operatorname{Im} h(n, \boldsymbol{p}))\phi'_n(p_0)\right) + k(p) \tag{38}$$

where $k(p) \in Q_\beta^{\text{triv}}$ as a consequence of the fact that the term in large round brackets in (38) is in $F\mathcal{S}$—a property which is easy enough, if rather tedious, to verify. This yields a complement of Q_β^{triv} and by inverse transform, one possible $\tilde{S}_\beta^{\text{triv}}$. A similar construction would work for non-trivial a.

Example 8. Let $\psi_c(p_0) = \exp\{Cp_0^2[p_0^2 - (2\pi/3\beta - \varepsilon)^2][p_0^2 - (2\pi/3\beta)^2]^{-1}\}$ for $|p_0| < 2\pi/3\beta$, zero elsewhere; and $0 < \varepsilon < 2\pi/3\beta$. Then for any $A > 0$ and $n \in \mathbb{Z}$ we can find C such that

$$\int \mathrm{d}p_0 \frac{|\psi_c(p_0 - 2\pi n/\beta)|^2}{(p_0^2 + A)} = \frac{2\pi}{\beta} \frac{1}{(2\pi n/\beta)^2 + A}. \tag{39}$$

Furthermore, for $A \in [m^2, \infty)$, C lies in a compact subset of $\mathbb{R}$. We now construct $\{\psi_{n,p}(p_0), \psi'_{n,p}(p_0)\}$ as follows: (i) $\psi_{0,p}(p_0) = \psi_c(p_0)$ with C chosen so that (39) holds with $n = 0$ and $A = (p^2 + m^2)$; (ii) for $n \geq 1$, $\psi_{n,p}(p_0) = [\psi_c(p_0 - 2\pi n/\beta) + \psi_c(p_0 + 2\pi n/\beta)]$ where we choose C such that (39) holds with $A = (p^2 + m^2)$ and n agreeing with the subscript of $\psi_{n,p}$; (iii) $\psi'_{n,p}(p_0) = \operatorname{sgn}(p_0)\psi_{n,r}(p_0)$. Then in a manner analogous to (38) we have for $h \in F\mathcal{S}$

$$\begin{aligned} h(p) &= \left(\sum_{n=0}^{\infty} (\operatorname{Re} h(n, p))\psi_{n,p}(p_0) + \mathrm{i} \sum_{n=1}^{\infty} (\operatorname{Im} h(n, p))\psi'_{n,p}(p_0) \right) + k(p) \\ &\equiv h_\psi(p) + k(p) \end{aligned} \tag{40}$$

where, this time, the checking that $h_\psi(p) \in F\mathcal{S}$ is somewhat more tedious than in example 7 due to the p dependence of $\psi_{n,p}(p_0)$. The complement R_β^{triv} of Q_β^{triv} that this procedure yields has the advantage that if $f, g \in R_\beta^{\text{triv}}$ then

$$\begin{aligned} \langle F^{-1}f, G_0 F^{-1}g \rangle_{L^2} &= \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \frac{\bar{f}g}{(p^2 + m^2)} \\ &= \frac{1}{(2\pi)^4} \frac{2\pi}{\beta} \sum_n \int \mathrm{d}^3 p \frac{\bar{f}(n, p)g(n, p)}{[(2\pi n/\beta)^2 + p^2 + m^2]} \\ &= \langle F^{-1}f, G_\beta^{\text{triv}} F^{-1}g \rangle, \end{aligned} \tag{41}$$

i.e. that the restriction of the covering space covariance to R_β^{triv} agrees with the periodic covariance given by the operator G_β^{triv}. If only we could condition onto R_β^{triv} then our entire objective would be achieved. But the argument fails at this last step.

Theorem 9. Let $\tilde{R}_\beta^{\text{triv}}$ be *any* complement of Q_β^{triv}. Then the covariance $G_0 - G_{\tilde{R}}$, where $G_{\tilde{R}}$ is the restriction of G_0 to $\tilde{R}_\beta^{\text{triv}}$, is not positive semidefinite.

Proof. Vectors in $\tilde{R}_\beta^{\text{triv}}$ are labelled by their values on $\mathbb{Z} \times \mathbb{R}^3$, the set on which all of Q_β^{triv} must vanish. Choose $h \in \tilde{R}_\beta^{\text{triv}}$ which is non-vanishing on $\{0\} \times \mathbb{R}^3$ and zero on $(\mathbb{Z}\backslash\{0\}) \times \mathbb{R}^3$. Now consider $g_\lambda(p) = \phi_0(\lambda p_0)h(p)$, $\lambda \geq 1$, where ϕ_0 is as in example 7. Since it agrees with h on $\mathbb{Z} \times \mathbb{R}^3$ it must differ from h by a vector $k_\lambda(p)$ in Q_β^{triv}:

$$g_\lambda(p) = \phi_0(\lambda p_0)h(p) = h(p) + k_\lambda(p). \tag{42}$$

Now there is some point $p \in \{0\} \times \mathbb{R}^3$ where $h(p) \neq 0$. By continuity, there is a ball b

of non-zero radius r centred on p wherein $h(p)$ is also non-vanishing. By choosing $\lambda > 2\pi(3\beta r)^{-1}$ we find

$$\begin{aligned}\langle F^{-1}g_\lambda(p), G_0F^{-1}g_\lambda(p)\rangle_{L^2} &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4\backslash b} d^4p \frac{|g_\lambda(p)|^2}{(p^2+m^2)} + \frac{1}{(2\pi)^4}\int_b d^4p \frac{|\phi_0(\lambda p_0)h(p)|^2}{(p^2+m^2)} \\ &< \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4\backslash b} d^4p \frac{|h(p)|^2}{(p^2+m^2)} + \frac{1}{(2\pi)^4}\int_b d^4p \frac{|h(p)|^2}{(p^2+m^2)} \\ &= \langle F^{-1}h, G_0F^{-1}h\rangle_{L^2}. \end{aligned} \tag{43}$$

Suppose, however, that $G_0 - G_{\tilde{R}}$ were positive semidefinite; then

$$\begin{aligned}&\langle F^{-1}g_\lambda(p), G_0F^{-1}g_\lambda(p)\rangle_{L^2} \\ &\quad= \langle F^{-1}(h+k_\lambda), (G_0-G_{\tilde{R}})F^{-1}(h+k_\lambda)\rangle_{L^2} + \langle F^{-1}(h+k_\lambda), G_{\tilde{R}}F^{-1}(h+k_\lambda)\rangle_{L^2} \\ &\quad= \langle F^{-1}k_\lambda, (G_0-G_{\tilde{R}})F^{-1}k_\lambda\rangle_{L^2} + \langle F^{-1}h, G_{\tilde{R}}F^{-1}h\rangle_{L^2} \\ &\quad= \langle F^{-1}k_\lambda, (G_0-G_{\tilde{R}})F^{-1}k_\lambda\rangle_{L^2} + \langle F^{-1}h, G_0F^{-1}h\rangle_{L^2} \\ &\quad> \langle F^{-1}h, G_0F^{-1}h\rangle_{L^2}, \end{aligned} \tag{44}$$

a contradiction which completes the proof.

The failure of conditioning which we have just demonstrated is vividly illustrated by a virulent incompatibility between the covariances for the covering theory and the automorphic theory. By letting $\lambda \to \infty$ in the construction of theorem 9 we see that $\langle F^{-1}g_\lambda, G_0F^{-1}g_\lambda\rangle \to 0$. On the other hand, since $\phi_0(0) = 1$, $\langle F^{-1}g_\lambda, G_\beta^{\text{triv}}F^{-1}g_\lambda\rangle$ remains constant, showing that there is no $1 < K < \infty$ such that $KG_0 \geq G_\beta^{\text{triv}}$. Conversely, for any $0 \neq h \in \tilde{T}_\beta^{\text{triv}}$ we clearly have $\langle h, G_\beta^{\text{triv}}h\rangle = 0$ so that we cannot arrange $KG_\beta^{\text{triv}} \geq G_0$ either. All of this is despite the known renormalisability of finite-temperature field theory which we showed in § 2.

Remark. Lest the reader be tempted to think otherwise, we note that there is no reason other than convenience for choosing the $\{\phi_n, \phi'_n\}$ to have disjoint supports in constructing complements of Q_β^a. In general this will not be the case (even if it were it would guarantee nothing), and vigilance has to be exercised. For example let $\chi_n(p_0) = \phi_n(p_0)$ for $|p_0| \leq n$, 1 for $|p_0| \in [n, n+1]$, and $\phi_{n+1}(p_0)$ for $|p_0| \geq n+1$. While each $\chi_n(p_0)$ is itself innocent enough, the family $\{\chi_n\}$ cannot yield a complementation of Q_β^{triv} in one dimension. (Hint: try to decompose ϕ_0 using the $\{\chi_n\}$, $\{\phi'_n\}$ and Q_β^{triv}.)

5. Boundary-value problems

In this section we will examine the connection between boundary-value problems and conditioning and compare these data with known facts from renormalised perturbation theory. Some of the following is an elaboration of results in GRSI, GRSII and Robinson (1971). This is unavoidable, if only from the need to establish notation. The formulation of the finite-temperature problem as a periodic boundary condition problem yields yet another approach to the material treated in the previous sections and again produces different results.

Our scenario in this section is the model of the free field given by the Gaussian random process indexed by $\mathscr{H}_{-1}$. The first fact we need, and the key to much of what follows, is the pre-Markoff property of $\mathscr{H}_{-1}$ (Nelson 1973b, GRSI). Let e_c be the projection in $\mathscr{H}_{-1}$ onto the subspace supported in the closed set C and $p_c = \mathbb{I} - e_c$. Let Λ be open in $\mathbb{R}^4$.

Lemma 10.

$$e_{\Lambda'} e_{\bar{\Lambda}} \mathscr{H}_{-1} \subset e_{\partial\Lambda} \mathscr{H}_{-1}.$$

Proof. Let h be the projection of $f \in e_{\bar{\Lambda}}\mathscr{H}_{-1}$ onto $e_{\Lambda'}\mathscr{H}_{-1}$. Then since $C_0^\infty(\Lambda'^0) \subset e_{\Lambda'}\mathscr{H}_{-1}$ we have for $g \in C_0^\infty(\Lambda'^0)$

$$\langle h, g\rangle_{L^2} = \langle h, (-\Delta + m^2)g\rangle_{-1} = \langle f, (-\Delta + m^2)g\rangle_{-1} = \langle f, g\rangle_{L^2} = 0 \tag{45}$$

since $f = 0$ as a distribution on $C_0^\infty(\Lambda'^0)$. Thus $\operatorname{supp} h \cap \Lambda'^0 = \varnothing$ and $h \in e_{\partial\Lambda}\mathscr{H}_{-1}$ completing the proof.

We note that since $\mathscr{H}_{-1}$ contains delta distributions concentrated on hypersurfaces of codimension one, as can be checked from (16), the above result is non-vacuous.

Lemma 11. If $f \in p_{\bar{\Lambda}}\mathscr{H}_{-1}$ then $\operatorname{supp} f \subset \Lambda'$.

Proof. Suppose $x_0 \in \Lambda$ was in $\operatorname{supp} f$. Then there would be a $g \in C_0^\infty(U(x_0))$ where $U(x_0)$ is a neighbourhood of x_0 contained in Λ, with $0 \neq \langle f, g\rangle_{L^2} = \langle f, (-\Delta + m^2)g\rangle_{-1}$. But $(-\Delta + m^2)g \in e_{\bar{\Lambda}}\mathscr{H}_{-1}$, so that $x_0 \notin \operatorname{supp} f$ completing the proof.

An important consequence of these two lemmas is the following theorem (cf GRSI).

Theorem 12. There is an orthogonal decomposition

$$\mathscr{H}_{-1} = e_{\partial\Lambda}\mathscr{H}_{-1} \oplus p_{\Lambda'}\mathscr{H}_{-1} \oplus p_{\bar{\Lambda}}\mathscr{H}_{-1}. \tag{46}$$

Proof. Since $\partial\Lambda = \Lambda' \cap \bar{\Lambda}$ we have $e_{\partial\Lambda}\mathscr{H}_{-1} = e_{\Lambda'}\mathscr{H}_{-1} \cap e_{\bar{\Lambda}}\mathscr{H}_{-1}$. Hence $p_{\partial\Lambda}\mathscr{H}_{-1} = p_{\Lambda'}\mathscr{H}_{-1} \oplus p_{\bar{\Lambda}}\mathscr{H}_{-1}$. Now $p_{\bar{\Lambda}}\mathscr{H}_{-1}$ is orthogonal to $p_{\Lambda'}\mathscr{H}_{-1}$ since any non-zero image of a projection from either to the other must lie in $e_{\partial\Lambda}\mathscr{H}_{-1}$ by lemma 11 and the pre-Markoff property; this is impossible completing the proof.

We now examine the dual space of $\mathscr{H}_{-1}$. As is well known, this is the space $\mathscr{H}_{+1}$, the closure of $\mathscr{S}$ in the inner product

$$\langle f, g\rangle_{+1} = \int d^4x\, [(\nabla f)\cdot(\nabla g) + m^2 fg]. \tag{47}$$

More precisely $\mathscr{H}_{+1}$ consists of those distributions which are in L^2 and whose first derivatives are also in L^2. The duality isomorphism $\mathscr{H}_{+1} \to \mathscr{H}_{-1}$ is just $(-\Delta + m^2)$ where $-\Delta$ has its distribution-theoretic meaning and the inverse is $(-\Delta + m^2)^{-1} = G_0$. Let e_c, p_c have the same significance in $\mathscr{H}_{+1}$ as they have in $\mathscr{H}_{-1}$ and let $V^\dagger$ denote $(-\Delta + m^2)^{-1} V \subset \mathscr{H}_{+1}$ for a closed subspace $V \subset \mathscr{H}_{-1}$.

Remark. $e_{\partial\Lambda}\mathscr{H}_{+1} = \varnothing$ since $\partial\Lambda$ has measure zero.

Lemma 13.

$$(e_{\partial\Lambda}\mathcal{H}_{-1})^{\dagger}=\{f\in\mathcal{H}_{+1}\colon(-\Delta+m^2)f=0 \text{ as a distribution on } \Lambda\cup\Lambda'^{0}\}. \quad (48)$$

Proof. Let $f\in(e_{\partial\Lambda}\mathcal{H}_{-1})^{\dagger}$ and $g\in C_0^{\infty}(\Lambda\cup\Lambda'^{0})$, then $\langle f,g\rangle_{+1}=\langle(-\Delta+m^2)f,g\rangle_{L^2}=0$ since $(-\Delta+m^2)f\in e_{\partial\Lambda}\mathcal{H}_{-1}$, completing the proof.

We define $H_0^1(\Lambda)$ to be the closure of $C_0^{\infty}(\Lambda)$ in the inner product (47).

Theorem 14. There is an orthogonal decomposition

$$\mathcal{H}_{+1}=(e_{\partial\Lambda}\mathcal{H}_{-1})^{\dagger}\oplus H_0^1(\Lambda)\oplus H_0^1(\Lambda'^{0}). \quad (49)$$

Proof. We have that $H_0^1(\Lambda)$ is orthogonal to $H_0^1(\Lambda'^0)$ since $\operatorname{supp} H_0^1(\Lambda)\cap \operatorname{supp} H_0^1(\Lambda'^0)=\partial\Lambda$ which is of measure zero. Let f be orthogonal to $H_0^1(\Lambda)\oplus H_0^1(\Lambda'^0)$, then if $g\in C_0^{\infty}(\Lambda\cup\Lambda'^0)$ we have $0=\langle f,g\rangle_{+1}=\langle(-\Delta+m^2)f,g\rangle_{L^2}$ so that $f\in(e_{\partial\Lambda}\mathcal{H}_{-1})^{\dagger}$ completing the proof.

Corollary 15. (i) $H_0^1(\Lambda)=(p_{\Lambda'}\mathcal{H}_{-1})^{\dagger}$; (ii) $p_{\Lambda'}\mathcal{H}_{-1}$ is the closure of $(-\Delta+m^2)C_0^{\infty}(\Lambda)$ in $\mathcal{H}_{-1}$.

Proof. $H_0^1(\Lambda)$ is the closure of $C_0^{\infty}(\Lambda)$ in $\mathcal{H}_{+1}$ and $(-\Delta+m^2)$ is the local duality isomorphism. Therefore the closure of $(-\Delta+m^2)C_0^{\infty}(\Lambda)$ in $\mathcal{H}_{-1}$ is a subspace of $\mathcal{H}_{-1}$ supported in $\bar{\Lambda}$ and orthogonal to $e_{\partial\Lambda}\mathcal{H}_{-1}$; and therefore also orthogonal to $p_{\bar{\Lambda}}\mathcal{H}_{-1}$. It is thus contained in $p_{\Lambda'}\mathcal{H}_{-1}$ and equal to $p_{\Lambda'}\mathcal{H}_{-1}$ since its orthogonal complement in $p_{\Lambda'}\mathcal{H}_{-1}$ consists of distributions f for which $0=\langle f,(-\Delta+m^2)g\rangle_{-1}=\langle f,g\rangle_{L^2}$ any $g\in C_0^{\infty}(\Lambda)$; whence $f\in e_{\partial\Lambda}\mathcal{H}_{-1}$ which implies $f=0$ completing the proof.

So we have decomposed both $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$ into direct sums of three orthogonal subspaces. Why bother? The answer is that (47), regarded as a bilinear form on $C_0^{\infty}(\Lambda)$, is just the form of the operator $(-\Delta+m^2)\restriction C_0^{\infty}(\Lambda)$ so $H_0^1(\Lambda)$ is the closure of the form domain and determines the Friedrichs extension of $(-\Delta+m^2)\restriction C_0^{\infty}(\Lambda)$ which is just $(-\Delta_{\Lambda}^{\mathrm{D}}+m^2)$, where $\Delta_{\Lambda}^{\mathrm{D}}$ is the Dirichlet extension of $\Delta\restriction C_0^{\infty}(\Lambda)$. Consequently, the restriction $(-\Delta+m^2)^{-1}\restriction p_{\Lambda'}\mathcal{H}_{-1}=G_{\Lambda}^{\mathrm{D}}$ is just the Dirichlet propagator for Λ and we conclude that the Dirichlet theory on Λ can be obtained by conditioning on the subspace $p_{\Lambda'}\mathcal{H}_{-1}\subset\mathcal{H}_{-1}$.

Having settled the Dirichlet problem, we move on to some other self-adjoint extensions of $-\Delta\restriction C_0^{\infty}(\Lambda)$ of interest to us. The first extension we consider is the free extension which is defined by specifying the propagator $G_0\restriction e_{\bar{\Lambda}}\mathcal{H}_{-1}$. Thus

$$G_{\Lambda}^{\mathrm{F}}=G_{\Lambda}^{\mathrm{D}}\oplus G_0\restriction e_{\partial\Lambda}\mathcal{H}_{-1} \quad (50)$$

and the domain of $(-\Delta_{\Lambda}^{\mathrm{F}}+m^2)$ is clearly $G_{\Lambda}^{\mathrm{F}}(L^2(\Lambda))$. The boundary condition for this operator is as complicated as the definition as a restriction makes its propagator simple. Generally it is

$$\partial f/\partial n=\Sigma_{\Lambda}f \qquad \text{on } \partial\Lambda \quad (51)$$

where Σ_{Λ} is a non-local pseudo-differential operator, even in the simplest case of Λ a half-space, as discussed in GRSII. As in the Dirichlet case, the free theory on Λ can be obtained by conditioning on $e_{\bar{\Lambda}}\mathcal{H}_{-1}$.

To define the Neumann problem we take the closure in the quadratic form (47) of the space $C^\infty(\bar\Lambda)$ and appeal to the standard result that a closed semibounded quadratic form yields a unique self-adjoint operator on a subdomain of the form domain (Reed and Simon 1972). That this is in line with our intuitive conception of the Neumann problem, can be checked by taking

$$f \in C^\infty(\bar\Lambda) \subset Q(-\Delta_\Lambda^N) \qquad \text{and} \qquad g \in C^\infty(\bar\Lambda) \cap D(-\Delta_\Lambda^N)$$

whence by integration by parts,

$$\int_\Lambda [(\nabla f)\cdot(\nabla g) + m^2 fg] = \int_{\partial\Lambda} f\,\partial g/\partial n + \int_\Lambda f(-\Delta + m^2)g. \tag{52}$$

The left-hand side of (52) is the Neumann quadratic form, and the volume integral on the right-hand side gives the action of the operator that the form defines, which must be equal to the left-hand side. Thus the surface integral must vanish and hence g must satisfy the usual Neumann condition

$$\partial g/\partial n = 0 \qquad \text{on } \partial\Lambda. \tag{53}$$

If we add the form

$$\int_{\partial\Lambda} f\sigma g \tag{54}$$

to the Neumann form, where $\sigma \in C^\infty(\partial\Lambda)$, then Robinson (1971) has shown that it is a relatively bounded perturbation of the Neumann form and hence determines a self-adjoint extension of $(-\Delta + m^2)\restriction C_0^\infty(\Lambda)$ by the KLMN theorem (Reed and Simon 1975). A similar manipulation to (52) shows that the corresponding boundary condition is (for $g \in C^\infty(\bar\Lambda) \cap D(-\Delta_\Lambda^\sigma)$)

$$\partial g/\partial n = \sigma g \qquad \text{on } \partial\Lambda. \tag{55}$$

If Λ is box-like, i.e. $\Lambda = \{x : |x_i| < L_i/2\}$, then we can take the closure of $C_P^\infty(\bar\Lambda)$ in (47) where $C_P^\infty(\bar\Lambda)$ is the subspace of $C^\infty(\bar\Lambda)$ whose restrictions to $\partial\Lambda$ satisfy

$$f(x) = f(x + L_i\hat r_i), \qquad P(\partial)f(x) = P(\partial)f(x + L_i\hat r_i) \qquad \forall P(\partial) \tag{56}$$

whenever both x and $x + L_i\hat r_i$ are in $\partial\Lambda$, where $\hat r_i$ is a unit vector in the ith direction. This yields the extension $(-\Delta_\Lambda^P + m^2)$ with periodic boundary conditions on Λ. We remark that antiperiodic and mixed problems, periodic in some directions and antiperiodic in others, may be treated in an analogous fashion.

Taking the Gaussian random processes indexed by $C_0^\infty(\Lambda)$ with covariances G_Λ^X or by the closures of $L^2(\Lambda)$ in the inner products given by the G_Λ^X, i.e. the inverses of the self-adjoint operators $(-\Delta_\Lambda^X + m^2)$ for the various different boundary conditions treated above, we get models for the free field with X-boundary conditions. There is a lattice of inequalities for the various G_Λ^X with obvious implications for conditioning and it is to this that we now turn our attention.

Consider two disjoint open regions Λ_1, Λ_2 with not necessarily disjoint closures and let $\Lambda = (\bar\Lambda_1 \cup \bar\Lambda_2)^0$. Denote the Neumann form on Λ by t_Λ^N and similarly for Λ_i. Then each t_i^N extends to a form on $L^2(\Lambda) = L^2(\Lambda_1) \oplus L^2(\Lambda_2)$ by prefacing it by a projection onto the appropriate factor. We have

$$t_\Lambda^N \subset t_{\Lambda_1}^N + t_{\Lambda_2}^N \tag{57}$$

since the right-hand side of (57) contains functions discontinuous across $\partial\Lambda_1 \cap \partial\Lambda_2$.

The right-hand side of (57) determines a self-adjoint operator which is an extension of $(-\Delta^{N}_{\Lambda_1}+m^2)\oplus(-\Delta^{N}_{\Lambda_2}+m^2)$ and which is, therefore, equal to it since $(-\Delta^{N}_{\Lambda_1}+m^2)\oplus(-\Lambda^{N}_{\Lambda_2}+m^2)$ is already self-adjoint. From the form domain inequality (57), we obtain the operator inequality (Kato 1966)

$$(-\Delta^{N}_{\Lambda}+m^2)\geq(-\Delta^{N}_{\Lambda_1}+m^2)\oplus(-\Delta^{N}_{\Lambda_2}+m^2) \tag{58}$$

which yields

$$G^{N}_{\Lambda}\leq G^{N}_{\Lambda_1}\oplus G^{N}_{\Lambda_2}. \tag{59}$$

Letting $\Lambda_2=\Lambda_1'^{0}$ we find (since $G^{N}_{\mathbf{R}^4}=G_0$)

$$G_0\leq G^{N}_{\Lambda_1}\oplus G^{N}_{\Lambda_2} \tag{60}$$

so that for any Λ we have

$$G^{F}_{\Lambda}\leq G^{N}_{\Lambda}. \tag{61}$$

If we repeat the above for the Dirichlet forms we find conversely

$$t^{D}_{\Lambda}\supset t^{D}_{\Lambda_1}+t^{D}_{\Lambda_2} \tag{62}$$

since functions on the right-hand side of (62) must vanish on $\partial\Lambda_1\cap\partial\Lambda_2$ whence

$$G^{D}_{\Lambda}\geq G^{D}_{\Lambda_1}\oplus G^{D}_{\Lambda_2} \tag{63}$$

and

$$G^{D}_{\Lambda}\leq G^{F}_{\Lambda} \tag{64}$$

consistent with our previous derivation. If $\sigma_1(x)\geq|\sigma_2(x)|$ for all $x\in\partial\Lambda$ then clearly

$$t^{\sigma_1}_{\Lambda}\geq t^{\sigma_2}_{\Lambda} \tag{65}$$

whence

$$G^{\sigma_1}_{\Lambda}\leq G^{\sigma_2}_{\Lambda}. \tag{66}$$

Also

$$\lim_{\substack{\sup|\sigma(x)|\to 0\\ x\in\partial\Lambda}} G^{\sigma}_{\Lambda}=G^{N}_{\Lambda} \tag{67}$$

and if $\sigma(x)>0$

$$\lim_{\substack{\inf|\sigma(x)|\to\infty\\ x\in\partial\Lambda}} G^{\sigma}_{\Lambda}=G^{D}_{\Lambda} \tag{68}$$

so that for $\sigma(x)\geq 0$

$$G^{D}_{\Lambda}\leq G^{\sigma}_{\Lambda}\leq G^{N}_{\Lambda}. \tag{69}$$

Finally, for box-like Λ consideration of the domain of t^{P}_{Λ} indicates that

$$G^{D}_{\Lambda}\leq G^{P}_{\Lambda}\leq G^{N}_{\Lambda}. \tag{70}$$

Thus we have established the lattice

$$\begin{array}{ccccccc} & \swarrow & G^{F}_{\Lambda} & \leftarrow & & \nwarrow & \\ G^{D}_{\Lambda} & \leftarrow & G^{\sigma_1}_{\Lambda} & \leftarrow & G^{\sigma_2}_{\Lambda} & \leftarrow & G^{N}_{\Lambda} \\ & \nwarrow & G^{P}_{\Lambda} & \leftarrow & & \swarrow & \end{array} \tag{71}$$

where the covariance at the tail of an arrow dominates the one at its point. Generally speaking 'most' self-adjoint extensions of $(-\Delta+m^2)\upharpoonright C_0^\infty(\Lambda)$ arising in practice give rise to covariances which dominate the Dirichlet and are dominated by the Neumann extensions but neither dominate or are dominated by each other. Thus the middle column of (71) is the most populated one and of the 'classical' boundary problems (i.e. D, N, σ, P) only the Dirichlet problem arises by conditioning on a subspace of $\mathscr{H}_{-1}$.

We now restrict our attention to box-like Λ and G_Λ^X obtainable by the method of images, to establish some inequalities of the form $K_\Lambda^{X,Y}G_\Lambda^X \geqslant G_\Lambda^Y$ on suitable domains. We can achieve most of them in one bound . . . and here it is (theorem III.4 of GRSII).

Theorem 16. For box-like Λ, $K_\Lambda^{\mathrm{F},X}$ exists for $X = \mathrm{D}, \mathrm{P}, \mathrm{N}$.

Proof. The case $X = \mathrm{D}$ is trivial by (64). For the other two cases, we can write

$$G_\Lambda^X = G_0 \sum_n U_n^X \tag{72}$$

where the U_n^X are the unitary translation and translation–reflection operators which realise G_Λ^X via the method of images. Thus for $f \in e_{\bar\Lambda}\mathscr{H}_{-1}$ we have

$$\begin{aligned}\langle f, G_\Lambda^X f\rangle_{L^2} &= \sum_n \langle f, U_n^X f\rangle_{-1} = \sum_n \langle f, e_{\bar\Lambda} U_n^X e_{\bar\Lambda} f\rangle_{-1} \\ &= \sum_n \langle f, e_{\bar\Lambda} e_{\bar\Lambda_n} U_n^X f\rangle_{-1}\end{aligned} \tag{73}$$

where Λ_n is the nth 'translate' of Λ as specified by U_n^X. By a further result in GRSI there is a bound $\|e_{\bar\Lambda} e_{\bar\Lambda_n}\| = \mathrm{O}(\exp(-m \operatorname{dist}(\Lambda, \Lambda_n)))$. Thus we have

$$\langle f, G_\Lambda^X f\rangle_{L^2} \leqslant \sum_n \|e_{\bar\Lambda} e_{\bar\Lambda_n}\| \, \|U_n^X\| \, \|f\|_{-1}^2 = K\|f\|_{-1}^2 = K\langle f, G_\Lambda^F f\rangle_{L^2} \tag{74}$$

completing the proof.

Corollary 17. G_Λ^X is a bounded quadratic form on $e_{\bar\Lambda}\mathscr{H}_{-1}$ for X and Λ as above.

Proof. Equation (74).

Proposition 18. For X, Y, Λ as above,

$$G_\Lambda^X - G_\Lambda^Y = (G_\Lambda^X - G_\Lambda^Y)\upharpoonright e_{\partial\Lambda}\mathscr{H}_{-1}. \tag{75}$$

Proof. To show (75) it is sufficient to show that $\langle f, (G_\Lambda^X - G_\Lambda^Y)g\rangle_{L^2} = 0$ for all $g \in p_{\Lambda'}\mathscr{H}_{-1}$. Now $(-\Delta+m^2)C_0^\infty(\Lambda)$ is dense in $p_{\Lambda'}\mathscr{H}_{-1}$ (corollary 15) and $G_\Lambda^X - G_\Lambda^Y$ satisfies $(-\Delta+m^2)(G_\Lambda^X - G_\Lambda^Y) = 0$ as a distribution, and is furthermore symmetric. So we have for $g = (-\Delta+m^2)h$, $h \in C_0^\infty(\Lambda)$

$$\langle f, (G_\Lambda^X - G_\Lambda^Y)g\rangle_{L^2} = \langle f, (G_\Lambda^X - G_\Lambda^Y)(-\Delta+m^2)h\rangle_{L^2} = \langle f, (-\Delta+m^2)(G_\Lambda^X - G_\Lambda^Y)h\rangle_{L^2} = 0 \tag{76}$$

completing the proof.

Corollary 19. For X, Λ as above

$$G_\Lambda^X = G_\Lambda^{\mathrm{D}} \oplus G_\Lambda^X \upharpoonright e_{\partial\Lambda}\mathscr{H}_{-1}. \tag{77}$$

Proof. Set $Y = \mathrm{D}$ in (75) and note that $G^{\mathrm{D}}_\Lambda \upharpoonright e_{\partial\Lambda}\mathscr{H}_{-1} = 0$.

The above conclusions considerably clarify the relationship between the various G^X_Λ, particularly for $X = \mathrm{N, D, F}$.

There remains the question of the nature of $G^{\mathrm{P}}_\Lambda \upharpoonright e_{\partial\Lambda}\mathscr{H}_{-1}$. For reasons that are explained in appendix 3 this is, in general, a very thorny problem, the basic reason for this being that while G^{P}_Λ is an image sum over G_0, the reverse is not the case. However, in the case of finite-temperature field theory, where $\partial\Lambda$ is particularly simple, consisting merely of $\{x \in \mathbb{R}^4 : |x_0| = \frac{1}{2}\beta\}$, a complete answer can be given. In fact we have the following theorem.

Theorem 20. If $\Lambda = \{x \in \mathbb{R}^4 : |x_0| < \frac{1}{2}\beta\}$ there is an orthogonal decomposition of $e_{\partial\Lambda}\mathscr{H}_{-1}$

$$e_{\partial\Lambda}\mathscr{H}_{-1} = A \oplus A^\perp \tag{78}$$

and G^{P}_Λ satisfies:

(1) $$G^{\mathrm{P}}_\Lambda \upharpoonright A = 0. \tag{79}$$

(2) There exists $K^{\mathrm{P,F}}_\Lambda$ such that

$$K^{\mathrm{P,F}}_\Lambda G^{\mathrm{P}}_\Lambda \upharpoonright A^\perp \geqslant G^{\mathrm{F}}_\Lambda \upharpoonright A^\perp. \tag{80}$$

For the proof see appendix 3.

5.1. Summary

Let us summarise what we have derived for N, F, P, D boundary conditions on box-like Λ (these conclusions are generally untrue for arbitrary boundary conditions and arbitrary Λ). Firstly, all the G^X_Λ coincide on $p_{\Lambda'}\mathscr{H}_{-1}$ and differences only exist on $e_{\partial\Lambda}\mathscr{H}_{-1}$. On $e_{\partial\Lambda}\mathscr{H}_{-1}$, G^{F}_Λ and G^{N}_Λ provide equivalent norms by (61) and (74), G^{D}_Λ is zero and, at least in the finite-temperature case, G^{P}_Λ splits $e_{\partial\Lambda}\mathscr{H}_{-1}$ into the orthogonal sum of A on which it vanishes and $A^\perp$ on which it provides a norm equivalent to that provided by G^{N}_Λ or G^{F}_Λ by (74) and (80). We remark that (74) and (80) contrast starkly with the analogous conclusions derived in § 4 for the periodic problem posed in a different setting.

The implications of all of this for conditioning and hence for renormalisation are evident. Thus all of the G^X_Λ field theories can be conditioned from the Neumann problem or multiples of the free problem. On the other hand this does not apply to the Dirichlet or periodic problems which each annihilate a non-zero subspace of $e_{\partial\Lambda}\mathscr{H}_{-1}$. Allowing ourselves the freedom given us by the existence of the $K^{X,Y}_\Lambda$, we derive the chain

$$\mathrm{N} \leftrightarrow \mathrm{F} \rightarrow \mathrm{P} \rightarrow \mathrm{D} \tag{81}$$

where an arrow points from a field theory to another which ought to be renormalisable by the same techniques as the first.

How does this compare with renormalised perturbation theory? The most complete treatment of interacting field theories in the presence of boundaries known to the author is that given by Symanzik (1981) who discusses the N and D problems. He finds that for the D case no specific surface counterterms need be included in the action, for a smooth boundary. This agrees with the implications of conditioning in so far as the D covariance can be conditioned from the imbedding theory. For the N case surface counterterms are required, and this agrees with the fact that G_Λ^{N} cannot be conditioned from G_0 although it disagrees with the fact that G_Λ^{N} may be conditioned from $K_\Lambda^{\mathrm{F,N}}G_0$. The reason why this should be the case is not presently understood although it may be linked to the fact that there are Λ (not box-like) for which there is no finite $K_\Lambda^{\mathrm{F,N}}$ (see GRSII).

One clear consequence of the chain (81) is that N counterterms ought to make finite the F theory. Since F boundary conditions are not in general use it is not possible to simply compare this prediction with known work. However, since G_Λ^{F} is perfectly easy to write down in momentum space, and the restriction to Λ is feasible in configuration space for sufficiently simple Λ, calculations can, in principle, be done and the prediction tested. That the result should require no surface counterterms at all (as we would expect from conditioning) is too much to expect, is shown by an early paper of Stueckelberg (1951) who in effect does a F boundary condition problem in QED_4 and finds surface divergences. The reasons for this are also unclear from a conditioning point of view.

For the P case the situation is rather more straightforward. We know that renormalisability follows from that of the covering space theory as demonstrated in § 2 and that the perturbation theory expressions do not distinguish between the theory factored from a covering space and its representation as a boundary value problem, the difference being chiefly one of domains and therefore only apparent at a more mathematical level than perturbation theory. This is strikingly illustrated by the existence of the inequalities $K_\Lambda^{X,Y}G_\Lambda^X \geq G_\Lambda^Y$ for $X, Y \in \{\mathrm{P}, \mathrm{F}\}$ on various subspaces of $e_{\partial\Lambda}\mathscr{H}_{-1}$ in the boundary value formulation, but their non-existence in the $|\Gamma| = \infty$ automorphic formulation with domain $\mathscr{S}$, the failure being attributable to the pathology of the $|\Gamma| = \infty$ situation, being absent if $|\Gamma| < \infty$.

6. Conclusion

From the previous four sections, and particularly from the paragraphs immediately preceding, we deduce that renormalisability and the implications of conditioning are broadly speaking in agreement. The theories which require surface counterterms appear higher up the 'conditioning chain' (81) than theories which do not. In particular, the closest agreement between conditioning and renormalisability has been obtained in precisely the place where it first suggested itself—namely multiply connected spaces with finite fundamental group (Banach 1980b).

Clearly though, conditioning is not the whole story where renormalisation is concerned, as evidenced by the discrepancies alluded to at the close of § 5, and in this regard two main areas for further work suggest themselves. The first would be a BHPZ-type of approach where one would attempt to derive a general result of the form 'conditioning plus 'something else' implies that renormalisability carries over from one theory to another'. There could well be a problem here since conditioning is not a concept that naturally finds a place within the circle of ideas normally associated with BHPZ.

Secondly, one could pursue the constructive work contained in this paper to its logical conclusion by taking a typical theory of interest, say $P(\phi)_2$ defined on $S^1 \times \mathbb{R}^1$ and demonstrating that the conditioning could be taken onto $S^1 \times \mathbb{R}^1$ (with an S^1 of say half the circumference of the first) in the limit of cut-off removal. This could be done in both the automorphic and boundary value formulations and the results compared. The author anticipates that such a programme could be carried to a successful conclusion using current constructive techniques and could well shed light on the tempting prospect of 'inverting the functional Fourier transform' referred to in § 3.

Unfortunately, as is obvious from the first page of this article, these matters must await the interest of other investigators.

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Appendix 1.

In order to show that the analytic interpolation of § 2 is well defined we must prove that a rational function on $\mathbb{C}^n$ (i.e. a ratio of multinomials) is uniquely specified by its values on $\mathbb{C}^{n-r}\times\mathbb{Z}^r$.

Theorem A1. Let $\{R_{i_1\ldots i_r}(z_1\ldots z_{n-r})\}$ be ratios of multinomials $N_{i_1\ldots i_r}/D_{i_1\ldots i_r}$ on $\mathbb{C}^{n-r}$. Then there is at most one rational function $R=N/D$ on $\mathbb{C}^n$ whose restrictions obey $R(z_1\ldots z_{n-r}, i_1\ldots i_r)=R_{i_1\ldots i_r}(z_1\ldots z_{n-r})$.

Proof. Suppose there were two such functions $R^i=N^i/D^i$, $i=1,2$. Consider the set of singularities of $R_{\{i\}}$. It is the set $D^{-1}_{\{i\}}(0)$ which is clearly of Lebesgue measure zero. The set $X=\bigcup_{\{\{i\}\}}D^{-1}_{\{i\}}(0)$ being a countable union of measure zero sets is also of measure zero and hence contains no balls. Therefore we can choose a countable set $\{x_j\}$, everywhere dense in $\mathbb{C}^{n-r}$, disjoint from X. By hypothesis

$$R^1(x_j, i_1\ldots i_r)=R^2(x_j, i_1\ldots i_r)=R_{i_1\ldots i_r}(x_j). \quad \text{(A1.1)}$$

Regarding $R^i(x_j, z_{n-r+1}\ldots z_n)$ as functions of z_{n-r+1} only, they are rational functions which, in one complex variable, have the unique factorisation property

$$R^i(x_j, z_{n-r+1}, i_2\ldots i_r)=N^i(x_j, z_{n-r+1}, i_2\ldots i_r)/D^i(x_j, z_{n-r+1}, i_2\ldots i_r) \quad \text{(A1.2)}$$

where N^i and D^i are polynomials with no common factor. Let

$$S^i(x_j, z_{n-r+1}, i_2\ldots i_r) = R^i(x_j, z_{n-r+1}, i_2\ldots i_r)D^1(x_j, z_{n-r+1}, i_2\ldots i_r)D^2(x_j, z_{n-r+1}, i_2\ldots i_r) \quad \text{(A1.3)}$$

then S^i are polynomials in z_{n-r+1} agreeing on $\mathbb{Z}$. By a corollary to Carlson's theorem (Titchmarsh 1932, § 5.81) $S^1=S^2$ everywhere, hence $R^1(z_1\ldots z_{n-r+1}, i_2\ldots i_r)=R^2(z_1\ldots z_{n-r+1}, i_2\ldots i_r)$ everywhere in $x_j\times\mathbb{C}$ where they are defined and since the $\{x_j\}$ are dense in $\mathbb{C}^{n-r}$, $R^1(z_1\ldots z_{n-r+1}, i_2\ldots i_r)=R^2(z_1\ldots z_{n-r+1}, i_2\ldots i_r)$ everywhere in $\mathbb{C}^{n-r+1}\times\mathbb{Z}^{r-1}$ where they are defined.

Now repeat the argument by finding a dense set $\{x_j'\}$ in $\mathbb{C}^{n-r+1}$ that avoids all the singularities of the extension of the $\{R_{i_1\ldots i_r}(z_1\ldots z_{n-r})\}$ to the $\{R_{i_2\ldots i_r}(z_1\ldots z_{n-r+1})\}\equiv\{R^i(z_1\ldots z_{n-r+1}, i_2\ldots i_r)\}$ and deriving a unique extension to $\mathbb{C}^{n-r+2}\times\mathbb{Z}^{r-2}$. A further $(r-2)$ repetitions of this algorithm yields the uniqueness we seek, completing the proof.

Let $\tilde{\mathcal{R}}$ be the field of rational functions on $\mathbb{C}^n$ and $\mathcal{R}_r$ the field of $\mathbb{Z}^r$ parametrised sets of rational functions on $\mathbb{C}^{n-r}$ possessing at least one extension to $\tilde{\mathcal{R}}$. Then by theorem A1, the restriction map $\theta: \tilde{\mathcal{R}}\to\mathcal{R}_r$, $\theta(R)=\{R_{i_1\ldots i_r}\}$ is an isomorphism so that any operator T on $\tilde{\mathcal{R}}$ has a unique image on $\mathcal{R}_r$ namely $\theta T\theta^{-1}$. This in particular applies to the Zimmermann $-t_p^{d(\gamma)}$ operators, enabling us to renormalise on $S^1\times\mathbb{R}^3$ if we can do so on $\mathbb{R}^4$.

Appendix 2.

We describe here the connection between the automorphic formalism and the bundle theoretical approach to fields on multiply connected spaces pursued by Isham and co-workers. Remarks similar to those which follow have been made in Chockalingham and Isham (1980). Of particular note in this regard is the convention in automorphic function theory of using left actions of a group on a topological space in contrast to the convention in bundle theory of using right actions (e.g. Husemoller 1966). Since we are more concerned here with the automorphic picture, our convention will be the left action. The bijection $g\to g^{-1}$ of course turns any left action into a right action and vice versa so we lose nothing either way.

Theorem A2. Let $\tilde{M}$ be the universal cover of a multiply connected space M with fundamental group Γ, covering projection $p:\tilde{M}\to M$, and $a(\Gamma)$ be a representation of Γ in a vector space V. Then there is a vector bundle (B,π,M) such that its cross sections are in bijective correspondence with V-valued functions on $\tilde{M}$ satisfying the automorphic condition $f(\gamma x)=a(\gamma)f(x)$.

Proof. Consider the product bundle $\tilde{B}=(\tilde{M}\times V,\tilde{\pi},\tilde{M})$ and define a left Γ-action on it by $\gamma:(\tilde{x},v)\to(\gamma\tilde{x},a(\gamma)v)$. The coset of $(\tilde{x},v)$ under this action is denoted $[\tilde{x},v]$ and we define B to be the collection $\{[\tilde{x},v]\}$. We make B into a bundle over M by giving it the quotient topology and equipping it with the projection map $\pi:B\to M$, $\pi([\tilde{x},v])=p(\tilde{x})=x$ which is manifestly well defined. B becomes a vector bundle under the scalar multiplication and addition laws in $\pi^{-1}(x)$; $\alpha[\tilde{x},v]=[\tilde{x},\alpha v]$, and $[\tilde{x}_1,v_1]+[\tilde{x}_2,v_2]=[\tilde{x}_1,v_1+a(\gamma^{-1})v_2]$ provided $\tilde{x}_2=\gamma\tilde{x}_1\in p^{-1}(x)$, which can easily be checked to be well defined and continuous. Let $\Pi:\tilde{B}\to B$ be the quotient map. Then we have $\Pi^{-1}(\pi^{-1}(x))=\bigcup_{\tilde{x}\in p^{-1}(x)}\tilde{\pi}^{-1}(\tilde{x})$ and since $\Pi^{-1}[\tilde{x},v]$ contains precisely one point out of each of the $\{\tilde{\pi}^{-1}(\tilde{x}'):\tilde{x}'\in p^{-1}(p\tilde{x})\}$ we see that the fibres of B are isomorphic to V by mapping $[\tilde{x},v]\to(\tilde{x}_0,a(\gamma)v)\to a(\gamma)v$ where $\tilde{x}_0=\gamma\tilde{x}$ for some fixed $\tilde{x}_0\in p^{-1}(p\tilde{x})$. This enables us to construct local trivialisations of B, $\psi_{\tilde{U}}$ by choosing open sets $\tilde{U}\subset\tilde{M}$ satisfying $\tilde{U}\cap\bigcup_{e\neq\gamma\in\Gamma}\gamma\tilde{U}=\varnothing$, $p(\tilde{U})=U$ and giving the isomorphisms $\tilde{\pi}^{-1}(\tilde{U})\approx\tilde{U}\times V\xrightarrow{p\times 1}U\times V\xrightarrow{\psi_{\tilde{U}}}\pi^{-1}(U)$. Given two open sets $U_1, U_2\subset M$ which have two open preimages $\tilde{U}_1,\tilde{U}_2\subset\tilde{M}$ satisfying the condition just mentioned, we construct the overlap function on $U_1\cap U_2$, $g_{\tilde{U}_1\tilde{U}_2}(x)=\psi_{\tilde{U}_1}^{-1}(x)\circ\psi_{\tilde{U}_2}(x)=a(\gamma(x))$ where $\gamma(x)$ is such that $\gamma\tilde{x}_2=\tilde{x}_1$ with $\tilde{x}_i$ being the unique points in $\tilde{M}$ such that $\tilde{x}_i\in\tilde{U}_i$ and $p(\tilde{x}_i)=x$. The set of all such U and their associated $\psi_{\tilde{U}}$ and $g_{\tilde{U}_1\tilde{U}_2}$ gives a coordinatisation of B.

Let ϕ be a section of B and suppose $\phi(x_0)=[\tilde{x}_0, v]$. Then if $\{(\tilde{x}_i, v_i): i=1,2\}\subset [\tilde{x}_0, v]$ there exists γ such that $\gamma\tilde{x}_1=\tilde{x}_2$ and $v_2=a(\gamma)v_1$ thus ϕ determines an automorphic section f of $\tilde{B}$ by $f(\gamma\tilde{x}_1)=f(\tilde{x}_2)=v_2=a(\gamma)v_1=a(\gamma)f(\tilde{x}_1)$ and since $\tilde{B}$ is a product bundle, f is a V-valued automorphic function on $\tilde{M}$. Conversely, since an automorphic function consists solely of equivalence classes of the quotient map Π, it determines a section of B completing the proof.

Appendix 3.

In this appendix we explore the nature of $G^{\mathrm{P}}_{\Lambda}\restriction e_{\partial\Lambda}\mathscr{H}_{-1}$ in greater depth for box-like Λ. Let $\partial\Lambda_i^{\pm}=\partial\Lambda\cap\{x\in\mathbb{R}^4: x_i=\pm\frac{1}{2}L_i\}$ be the $\pm i$th face of $\partial\Lambda$. Clearly $\partial\Lambda_i^{\pm}$ is closed, $\partial\Lambda=\bigcup_{i,\pm}\partial\Lambda_i^{\pm}$, $e_{\partial\Lambda_i^{\pm}}\mathscr{H}_{-1}$ is a closed subspace of $\mathscr{H}_{-1}$ and $\Sigma_{i,\pm}\, e_{\partial\Lambda_i^{\pm}}\mathscr{H}_{-1}\subset e_{\partial\Lambda}\mathscr{H}_{-1}$.

Consider a class $\mathscr{P}$ of partitions of unity in $\mathbb{R}^4$, $\{\varphi_i\}$, having the following properties.

(1) $\{\varphi_i\}$ is locally finite and each φ_i has compact support.

(2) For each 'corner' of $\partial\Lambda$, the non-empty intersection of four distinct $\partial\Lambda_i^{\pm}$, there is a φ_i whose support contains no other corner and such that $\varphi_i=1$ in a neighbourhood of that corner.

(3) For each 'one-edge' of $\partial\Lambda$, the non-empty intersection of three distinct $\partial\Lambda_i^{\pm}$, there is a subset of the $\{\varphi_i\}$, whose supports intersect no other one-edge and whose sum, with the inclusion of the two φ_j associated with the two corners included in the one-edge if it is compact, is identically one in a neighbourhood of that one-edge.

(4) For each 'two-edge' of $\partial\Lambda$, the non-empty intersection of two distinct $\partial\Lambda_i^{\pm}$, there is a subset of the $\{\varphi_i\}$, whose supports intersect no other two-edge and whose sum, with the inclusion of any $\{\varphi_j\}$ associated with any corners or one-edges included in the two-edge, is identically one in a neighbourhood of that two-edge.

(5) The analogous construction for 'three-edges' (which are, of course, just the faces $\partial\Lambda_i^{\pm}$).

Proposition A3. $\Sigma_{i,\pm}\, e_{\partial\Lambda_i^{\pm}}\mathscr{H}_{-1}$ is a direct sum.

Proof. Suppose $\psi\in\Sigma_{i,\pm}\, e_{\partial\Lambda_i^{\pm}}\mathscr{H}_{-1}$ and we can write

$$\psi=\sum_{i,\pm} f_i^{\pm}=\sum_{i,\pm} g_i^{\pm}; \tag{A3.1}$$

then we have to show that $f_i^{\pm}=g_i^{\pm}$ for each index $(i,\pm)$. From (A3.1) we have that

$$h_1^{+}=f_1^{+}-g_1^{+}=\sum_{(i,\pm)\neq(1,+)} g_i^{\pm}-f_i^{\pm} \tag{A3.2}$$

and from (A3.2) we see that h_1^{+} is supported in $\partial\Lambda_1^{+}\cap\bigcup_{(i,\pm)\neq(1,+)}\partial\Lambda_i^{\pm}$ which is a non-disjoint collection of two-edges, one-edges and corners. Now let $\{\varphi_j\}\in\mathscr{P}$ and write

$$h_1^{+}=\sum_{jkl}\varphi_{Aj}h_1^{+}+\varphi_{Bk}h_1^{+}+\varphi_{Cl}h_1^{+} \tag{A3.3}$$

where the $\{\varphi_{Aj}\}$ are those members of $\{\varphi_j\}$ associated with the two-edges in $\operatorname{supp} h_1^{+}$, the $\{\varphi_{Bk}\}$ are those members of $\{\varphi_j\}$ associated with the one-edges in $\operatorname{supp} h_1^{+}$ and $\{\varphi_{Cl}\}$ are associated with the corners in $\operatorname{supp} h_1^{+}$. Pick one particular $\varphi_{Aj}h_1^{+}$ term and consider its Fourier transform. We have that $\varphi_{Aj}h_1^{+}$ is compactly supported on a hyperplane of dimension two, therefore $F(\varphi_{Aj}h_1^{+})$ is (modulo a factor e^{ipx}) a polynomially bounded entire function of type zero in at least two momentum variables by the Paley–Wiener theorem. By a theorem of Bernstein (Boas 1954, theorem 6.2.13) we must have that $|F(\varphi_{Aj}h_1^+)|^2$ *is* a polynomial in the modulus of each of these two momentum variables and thus, for its $\mathscr{H}_{-1}$ norm to converge, must be identically zero. So $\varphi_{Aj}h_1^+=0$. Since this result holds for all the φ_{Aj} in all $\{\varphi_j\}\in\mathscr{P}$ we see that h_1^+ must be supported on the one-edges in $\partial\Lambda_1^+$. We now repeat the same reasoning for the $\varphi_{Bk}h_1^+$, the Paley–Wiener–Bernstein argument holding for three momentum variables; and then again for the $\varphi_{Cl}h_1^+$. We conclude that h_1^+ has no support and so is zero. Repeating the derivation for all other indices $(i,\pm)$ completes the proof.

We emphasise here that we have not said whether or not $\Sigma_{i,\pm}e_{\partial\Lambda_i^\pm}\mathscr{H}_{-1}$ spans $e_{\partial\Lambda}\mathscr{H}_{-1}$. There may well be distributions in $e_{\partial\Lambda}\mathscr{H}_{-1}$ which cannot be written as a sum of distributions supported on the individual faces. The reader familiar with the flavour of modern boundary-value problem theory will not be startled by such caution.

Next, a little abstract nonsense. We recall that a quadratic form on a Hilbert space H is coercive if it satisfies $G(x,x)\geqslant K\|x\|^2$ for all vectors $x\in H$ and some $K>0$.

Lemma A4. Let A be a closed subspace of a Hilbert space H and G a non-negative quadratic form which satisfies $G(x,x)=0$ for all $x\in A$. Then G is coercive on $A^\perp$ if and only if it is coercive on any other (topological) complement of A.

Proof. Let S be a topological complement of A in H, G be coercive on $A^\perp$ (with constant K), and let $x=a+b=u+v$ where $a,u\in A$, $b\in A^\perp$ and $v\in S$. Then the map $\alpha: A^\perp\to S: b\to v=b+(a-u)$ is one–one and onto since S is a complement, and continuous since S is a topological complement so there is a K_1 such that $\|b\|\geqslant K_1\|v\|$. Since G is non-negative, it satisfies a Cauchy–Schwartz inequality, whence $G(x,u)=0$ if $u\in A$. Therefore

$$G(v,v)=G(b+(a-u),b+(a-u))=G(b,b)\geqslant K\|b\|^2\geqslant KK_1^2\|v\|^2 \qquad \text{(A3.4)}$$

and G is coercive on S. Reversing the roles of $A^\perp$ and S in the above argument gives the reverse implication, completing the proof.

We define convenient subspaces A and S in $\Sigma_{i,\pm}e_{\partial\Lambda_i^\pm}\mathscr{H}_{-1}$ as follows. Pick i and define the projections α_i and σ_i on $e_{\partial\Lambda_i^+}\mathscr{H}_{-1}\oplus e_{\partial\Lambda_i^-}\mathscr{H}_{-1}$ by

$$\begin{aligned}\alpha_i(f)&=\alpha_i(f^+(\ldots x_i\ldots)+f^-(\ldots x_i\ldots))\\&=\tfrac{1}{2}\{f^+(\ldots x_i\ldots)+f^-(\ldots x_i\ldots)-f^+(\ldots -x_i\ldots)-f^-(\ldots -x_i\ldots)\}\end{aligned} \qquad \text{(A3.5)}$$

$$\begin{aligned}\sigma_i(f)&=\sigma_i(f^+(\ldots x_i\ldots)+f^-(\ldots x_i\ldots))\\&=\tfrac{1}{2}\{f^+(\ldots x_i\ldots)+f^-(\ldots -x_i\ldots)+f^+(\ldots -x_i\ldots)+f^-(\ldots -x_i\ldots)\}.\end{aligned} \qquad \text{(A3.6)}$$

Here $f^\pm$ are the components of f supported in $\partial\Lambda_i^\pm$ and $f^\pm(\ldots -x_i\ldots)$ are the distributions obtained by reflecting in the $x_i=0$ hyperplane. Clearly $\|\alpha_i\|=\|\sigma_i\|=1$ and $\alpha_i+\sigma_i=\mathbb{I}\upharpoonright(e_{\partial\Lambda_i^+}\mathscr{H}_{-1}\oplus e_{\partial\Lambda_i^-}\mathscr{H}_{-1})$. Defining $\alpha=\Sigma_i\alpha_i$, $\sigma=\Sigma_i\sigma_i$ we construct the subspaces A and S by

$$A=\operatorname{Im}\alpha=\operatorname{Ker}\sigma \qquad S=\operatorname{Im}\sigma=\operatorname{Ker}\alpha. \qquad \text{(A3.7)}$$

Now if $f\in A$ then f is an antiperiodic boundary form hence $\Sigma_n U_n^{\mathrm{P}}f=0$ where the U_n^{P} are the translation operators which implement periodic boundary conditions, thus $\langle f, G_\Lambda^{\mathrm{P}}f\rangle_{L^2}=0$ and by lemma A4, coercivity on $A^\perp$ is equivalent to coercivity on

S provided $\Sigma_{i,\pm} e_{\partial\Lambda_i^{\mp}}\mathcal{H}_{-1}$ spans $e_{\partial\Lambda}\mathcal{H}_{-1}$. This is certainly the case for finite-temperature field theory posed as a boundary value problem since $\Lambda = \{x \in \mathbb{R}^4 : |x_0| < \frac{1}{2}\beta\}$ has a boundary consisting of a disjoint union of the faces $\partial\Lambda_0^+ = \{x \in \mathbb{R}^4 : x_0 = \frac{1}{2}\beta\} \cup \partial\Lambda_0^- = \{x \in \mathbb{R}^4 : x_0 = -\frac{1}{2}\beta\}$.

Proof of theorem 20. Let $f \in e_{\partial\Lambda_0^{\mp}}\mathcal{H}_{-1}$. Introduce a partition of unity $\{\varphi_i\} \in \mathcal{P}$ and write $f = \Sigma f_i$ where $f_i = \varphi_i f$. Then we can apply the Paley–Wiener–Bernstein argument of proposition A3 to deduce that the Fourier transform of f_i is of the form $e^{ip_0\beta/2}\tilde{f}_i(p_1, p_2, p_3)$. Adding, we find that the same is true for f. Thus if $g \in S$, its transform is of the form $\cos(\beta p_0/2)\tilde{g}(\boldsymbol{p})$. We can thus evaluate $\|g\|_{-1}^2$ and $\langle g, G_\Lambda^{\mathrm{P}} g\rangle_{L^2}$ explicitly. We find

$$\|g\|_{-1}^2 = \langle g, G_0 g\rangle_{L^2} = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p \frac{\cos^2(\beta p_0/2)|\tilde{g}(\boldsymbol{p})|^2}{p_0^2 + \boldsymbol{p}^2 + m^2}$$

$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^3 p |\tilde{g}(\boldsymbol{p})|^2 \Phi_1[(\boldsymbol{p}^2 + m^2)^{1/2}] \qquad \text{(A3.8)}$$

$$\langle g, G_\Lambda^{\mathrm{P}} g\rangle_{L^2} = \frac{1}{(2\pi)^4}\int \mathrm{d}^3 p \frac{2\pi}{\beta}\sum_n \frac{|\tilde{g}(\boldsymbol{p})|^2}{(2\pi n/\beta)^2 + \boldsymbol{p}^2 + m^2}$$

$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^3 p |\tilde{g}(\boldsymbol{p})|^2 \Phi_2[(\boldsymbol{p}^2 + m^2)^{1/2}] \qquad \text{(A3.9)}$$

where

$$\Phi_1(x) = \frac{\pi}{2}\frac{(1 + \exp(-\beta x))}{x} \qquad \Phi_2(x) = \pi\frac{\coth(\beta x/2)}{x}. \qquad \text{(A3.10)}$$

Since Φ_1 and Φ_2 have the same asymptotic form and are continuous on $(0, \infty)$ we can clearly arrange a constant $K(m^2)$ such that $K(m^2)\Phi_2(x) \geq \Phi_1(x)$ for $x \geq m > 0$ and hence that the integrand in (A3.9) dominates the integrand in (A3.8) pointwise, leading to $K(m^2)\langle g, G_\Lambda^{\mathrm{P}} g\rangle_{L^2} \geq \langle g, g\rangle_{-1}$ completing the proof of coercivity.

To conclude, we remark that even putting aside the issue of whether $\Sigma_{i,\pm} e_{\partial\Lambda_i^{\mp}}\mathcal{H}_{-1}$ spans $e_{\partial\Lambda}\mathcal{H}_{-1}$ or not it is evident that the above proof would have become considerably more difficult had we had more than one pair of faces to deal with in $\partial\Lambda$.

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